

THE YOGA OF SCHEMIC GROTHENDIECK RINGS, A TOPOS-THEORETICAL APPROACH

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ABSTRACT. We propose a suitable substitute for the classical Grothendieck ring of an algebraically closed field, in which any quasi-projective scheme is represented, while maintaining its non-reduced structure. This yields a more subtle invariant, called the schemic Grothendieck ring, in which we can formulate a form of integration resembling Kontsevich's motivic integration via arc schemes. Whereas the original construction was via definability, we have translated in this paper everything into a topos-theoretic framework.

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1. INTRODUCTION

Kontsevich [9] has formulated a general integration technique on (smooth) schemes over an algebraically closed field κ , modeled on p -adic integration and called *motivic integration*. This was then extended by Denef and Loeser [1, 2, 3] to achieve *motivic rationality*, by which they mean the fact that the rationality of a certain generating series from geometry or number-theory, like the Igusa-zeta series, is “motivated” by the rationality of its motivic counterpart which specializes to the given classical series via some multiplicative function (counting function, Euler characteristic, ...). The two main ingredients of

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this construction are the Grothendieck ring of varieties over κ (see below), in which the integration takes its values, and the *arc space* $L(X)$ of a variety X , that is to say, the reduced Hilbert scheme classifying all arcs $\text{Spec } \kappa[[\xi]] \rightarrow X$. Our aim is to extend this by replacing varieties by schemes, in such a way that killing the nilpotent structure reverts to the old theory. The classical Grothendieck ring $\mathbf{Gr}(\mathcal{V}\text{ar}_\kappa)$ of an algebraically closed field κ is designed to encode both combinatorial and geometric properties of varieties. It is defined as the quotient of the free Abelian group on varieties over κ , modulo the relations $[X] - [X']$, if $X \cong X'$, and

$$(1) \quad [X] = [X - Y] + [Y],$$

if Y is a closed subvariety, for Y, X, X' varieties (=reduced, separated schemes of finite type over κ). We will refer to the former relations as *isomorphism relations* and to the latter as *scissor relations*, in the sense that we “cut out Y from X .” Multiplication on $\mathbf{Gr}(\mathcal{V}\text{ar}_\kappa)$ is then induced by the fiber product. In sum, the three main ingredients for building the Grothendieck ring are: scissor relations, isomorphism relations, and products. Only the former causes problems if one wants to extend the construction of the Grothendieck ring from varieties to arbitrary finitely generated schemes. Put bluntly, we cannot cut a scheme in two, as there is no notion of a scheme-theoretic complement, and so we ask what new objects we should add to make this work. Let us call these new objects tentatively *motives*, in the sense that their existence is motivated by a combinatorial necessity. There are now two approaches to construct these motives.

The first one, discussed at length in [14], is based on definability, and was the original approach. The point of departure is the representation of a scheme by an equational (first-order) formula modulo the theory of local Artin algebras. The logical operations of conjunction, disjunction, and negation are then used to form the required new objects: motives arise as Boolean combinations of schemes. Scissor relations are now easily expressed in this formalism, whereas products are given by conjunction with respect to distinct variables, and isomorphism relations are phrased in terms of definable isomorphisms, cumulating in the construction of the *schemic Grothendieck ring* over κ . To obtain geometrically more significant motives, one is forced not only to introduce quantifiers, but also to resort to some infinitary logic via formularies, leading to larger Grothendieck rings, all of which still admit a natural homomorphism onto the classical Grothendieck ring. To define the analogue of arc spaces, one obtains arc formulae by interpreting the theory of a local Artin algebra in that of its residue field. The resulting arc operator is compatible with Boolean combinations and hence induces an endomorphism on the Grothendieck rings. This was the first striking success of the new theory: no such operation holds in the classical Grothendieck ring. Other main advantages of this approach are (i) the presence of negation, allowing one to “cut up” a scheme into motives, and (ii) the uniformity inherent in model-theory, allowing one to use parameters, and hence to work over an arbitrary base ring rather than just an algebraically closed field. The main disadvantage stems from non-functoriality, in particular when dealing with morphisms. Nonetheless, the theory has been applied in [14] with success to establish motivic rationality in certain cases, even the, thus far illusive, positive characteristic case.

However, soon after writing down this version, I came to realize that it might be beneficial to sacrifice definability for functoriality. In this approach, which forms the content of this paper and is essentially topos-theoretical, schemes are viewed as (contravariant) functors. Traditionally, one views them as functors, called *representable functors*, from the category of κ -algebras to the category of sets, but the power of the present approach comes from narrowing down the former category to that of *fat points*, consisting only of one-point

schemes over κ . Thus, given a scheme X of finite type over κ and a fat point \mathfrak{z} , we let $X(\mathfrak{z})$ be the set of all \mathfrak{z} -rational points, that is to say, morphisms $\mathfrak{z} \rightarrow X$. The functor $\mathfrak{z} \mapsto X(\mathfrak{z})$ now determines the scheme X uniquely. Motives are then certain subfunctors of these representable functors, with morphisms between them given as natural transformations. Since these functors take values in the category of sets, all set-theoretic operations are available to us, such as union, intersection, and complement. However, complementation does not behave functorially, and so motives now only form a lattice, leading to the notion of a *motivic site*: apart from a Grothendieck topology inherited from the Zariski topology on the schemes, we also require a categorical lattice structure in order to formulate scissor relations. Defining multiplication by means of fiber products, we thus get the Grothendieck ring of a motivic site. Among the many tools from category theory and topos theory we can now resort to, adjunction takes a primary place: it allows us, for instance, to define, without much effort, arc schemes, which act again nicely on the corresponding Grothendieck rings.

Let me now briefly discuss in more detail the content of the present paper. In §2, we discuss the functors that will play the role of motives. Borrowing terminology from topos theory, on the category of fat points, a subfunctor \mathfrak{X} of a representable functor given by a scheme X is called a *sieve* on X , and X is called an *ambient space* of \mathfrak{X} . We may do this over an arbitrary Noetherian base scheme V , provided it is also Jacobson. For the sake of this introduction, I will only treat the case of greatest interest to us, namely, when V is the spectrum of an algebraically closed field κ . A morphism of sieves is in principle any natural transformation, but often such a morphism extends to a morphism of the ambient spaces, in which case we call it *algebraic*. We turn this into a true topos in §3, by defining an admissible open of a sieve \mathfrak{X} to be its restriction to an open in its ambient space. We define a *global section* on a sieve \mathfrak{X} to be any morphism into the affine line. We establish an acyclicity result for global sections, allowing us to define the *structure sheaf* $\mathcal{O}_{\mathfrak{X}}$ of \mathfrak{X} .

In the next four sections, §§4–7, we introduce the Grothendieck ring of a motivic site, and discuss the three main cases. As already mentioned, a motivic site is for each scheme, a choice of lattice of sieves on that scheme, called the *motives* of the site. The associated Grothendieck ring is then defined as the free Abelian group on motives in the site modulo the isomorphism relations and the scissor relations, where the latter take the lattice form

$$(2) \quad [\mathfrak{X}] + [\mathfrak{Y}] = [\mathfrak{X} \cup \mathfrak{Y}] + [\mathfrak{X} \cap \mathfrak{Y}],$$

for any two motives \mathfrak{X} and \mathfrak{Y} on the same ambient space. The first motivic site of interest consists of the *schemic* motives, given on each scheme as the lattice of its closed subschemes (viewed as representable subfunctors). The resulting Grothendieck ring is too coarse, as it is freely generated as an additive group by the classes of irreducible schemic motives (Theorem 5.7). A larger, more interesting site is given by the sub-schemic motives, where we call a sieve \mathfrak{X} on X *sub-schemic*, if there is a morphism $\varphi: Y \rightarrow X$ such that at each fat point \mathfrak{z} , the set $\mathfrak{X}(\mathfrak{z})$ consists of all \mathfrak{z} -rational points $\mathfrak{z} \rightarrow X$ that factor through φ , that is to say, $\mathfrak{X}(\mathfrak{z})$ is the image of the induced map $\varphi(\mathfrak{z}): Y(\mathfrak{z}) \rightarrow X(\mathfrak{z})$. Any locally closed subscheme is sub-schemic, so that in the corresponding Grothendieck ring, we may express the class of any separated scheme in terms of classes of affine schemes. Moreover, any morphism of sieves with domain a sub-schemic motif is algebraic (Theorem 3.7), from which it follows that the sub-schemic Grothendieck ring admits a natural homomorphism into the classical Grothendieck ring.

Whereas in general the complement of a sieve is no longer a sieve (as functoriality fails), this does hold for any open subscheme. However, such a complement is in general no longer sub-schemic, but only what we will call a *formal motif*, that is to say, a sieve \mathfrak{X} that can be approximated by sub-schemic submotives in the sense that for each fat point

\mathfrak{z} , one of its sub-schematic approximations has the same \mathfrak{z} -rational points as \mathfrak{X} . In case of an open $U = X - Y$, the complement is represented by the formal completion \hat{X}_Y , whose approximations are the *jet spaces* $J_Y^n X := \text{Spec}(\mathcal{O}_X/\mathcal{I}_Y^n)$. A morphism in the site of formal motives $\mathbb{F}\text{orm}_\kappa$ is approximated by algebraic morphisms, and therefore, the ensuing Grothendieck ring $\mathbf{Gr}(\mathbb{F}\text{orm}_\kappa)$ still admits a canonical homomorphism onto the classical Grothendieck ring $\mathbf{Gr}(\text{Var}_\kappa)$.

In §8, we discuss *adjunction* of motivic sites over different base schemes. Formally, this consists of a pair of functors $\eta: \mathbb{F}\text{at}_W \rightarrow \mathbb{F}\text{at}_V$ and $\nabla_\eta: \mathbf{Sieve}_V \rightarrow \mathbf{Sieve}_W$ such that

$$\mathfrak{X}(\eta(\mathfrak{z})) = (\nabla_\eta \mathfrak{X})(\mathfrak{z})$$

for any V -sieve \mathfrak{X} and any W -fat point \mathfrak{z} . We call such an adjunction (sub-)schematic or formal, if $\nabla_\eta X$ is respectively (sub-)schematic or formal, for any scheme X . Under this assumption, the adjunction induces a homomorphism from the corresponding Grothendieck ring over V to the Grothendieck ring over W . Whenever we have a morphism $f: V \rightarrow W$ of finite type, we obtain an adjunction given by the pair $(f_*, f^* = \nabla_{f*})$, called *augmentation*, where f_* means restriction of scalars and f^* means extension of scalars via f . If f is moreover finite and flat, then we can go the other way, called *diminution*, via the pair (f^*, ∇_{f*}) . Both adjunctions are schematic, and they are related by the projection formula

$$\nabla_{\tilde{f}*} \nabla_{\tilde{h}*} = \nabla_{h*} \nabla_{f*}$$

where $h: \tilde{V} \rightarrow V$ is of finite type, and \tilde{f} and \tilde{h} are the corresponding base changes of f and h with respect to the other (Theorem 8.11). Applying diminution to a rational point $a \in X(\kappa)$, we may define for each morphism $Y \rightarrow X$ and each motif \mathfrak{Y} on Y , the *specialization* \mathfrak{Y}_a . In this way, \mathfrak{Y} becomes a family of motives. Another interesting application of adjunction is via the action of Frobenius \mathbf{F} in characteristic $p > 0$. If we let $\mathbf{F}(\mathfrak{z})$ be the fat point with coordinate ring the p -th powers of elements in the coordinate ring of \mathfrak{z} , then this yields an adjunction $(\mathbf{F}, \nabla_{\mathbf{F}})$ which is only sub-schematic: $\nabla_{\mathbf{F}} Y$ is given as the image of the relative Frobenius on Y (locally via some embedding in affine space; see Theorem 8.14 for a precise formulation).

In §9, we apply the adjunction theory to the special case of the structure morphism $j: \mathfrak{z} \rightarrow \text{Spec } \kappa$ of a fat point, and we define the *arc functor* $\nabla_{\mathfrak{z}}$ as the composition of augmentation and diminution along j , that is to say,

$$\nabla_{\mathfrak{z}} := \nabla_{j*} \circ \nabla_{j*}.$$

This corresponds in the special case that $\mathfrak{z} = \mathfrak{l}_n := \text{Spec } \kappa[\xi]/(\xi^n)$ to the classical truncated arc space L_n through the formula

$$L_n(X) = (\nabla_{\mathfrak{l}_n} X)^{\text{red}}$$

for any variety X . However, it should be noted that $\nabla_{\mathfrak{z}}$ does not commute with taking reductions, so that even if X and X' have the same underlying variety, they will in general have different arc schemes $\nabla_{\mathfrak{z}} X$ and $\nabla_{\mathfrak{z}} X'$, even possibly of different dimension (see §10). In fact, the dimension of the arc scheme of a fat point over itself is an intriguing invariant.

By the general theory of adjunction, $\nabla_{\mathfrak{z}}$ is an endomorphism on each Grothendieck ring. Arcs behave well over smooth varieties, as in the classical case (Theorem 9.10): the canonical morphism $\nabla_{\mathfrak{z}} X \rightarrow X$ is a locally trivial fibration over the non-singular locus of X , with general fiber some affine space. Hence, in the smooth case, $[\nabla_{\mathfrak{z}} X] = [X] \mathbb{L}^{l(d-1)}$, where d and l are respectively the dimension of X and the length of (the coordinate ring of) \mathfrak{z} , and where $\mathbb{L} := [\mathbb{A}_\kappa^1]$ is the *Lefschetz class*.

Using the formalism of adjunction, we discuss some variants of arcs: *deformed arcs* in §11, and *extendable arcs* in §13. For the definition of the latter, we discuss in §12 a compactification of the category of fat points, the category of *limit points*, given as direct limits of fat points (e.g., the formal completion \hat{Y}_P). Although we can extend the notion of arcs to any limit point, the corresponding arc scheme is no longer of finite type.

In §14, we discuss some of the motivic series that can now be defined using this formalism. As already mentioned, since they or their classical variants specialize to generating series that are known to be rational, we expect them to be already rational over the formal Grothendieck ring, or rather, over its localization $\mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}$; this is called *motivic rationality*. Let us briefly describe each of these series here.

Igusa: with $j_n := J_P^n Y$ the n -th jet of a closed point P on a scheme Y , and X any scheme of dimension d , we set

$$\mathrm{Igu}_{X_{\mathrm{mot}}}^{(Y,P)}(t) := \sum_{n=1}^{\infty} \mathbb{L}^{-d\ell(j_n)} [\nabla_{j_n} X] t^n$$

Hilbert: with (Y, P) and j_n as above, we set

$$\mathrm{Hilb}_{(Y,P)}^{\mathrm{mot}} := \sum_{n=1}^{\infty} [j_n] t^n$$

Hilbert-Kunz: with κ of characteristic $p > 0$, with $Y \subseteq X$ a closed subscheme, and with $Y_X^{[n]}$ the n -th Frobenius transform of Y (defined by the p^n -th powers of the defining equations of Y), we set

$$\mathrm{HK}_Y^{\mathrm{mot}}(X) := \sum_{n=1}^{\infty} [Y_X^{[n]}] t^n$$

Milnor: with η_n the n -th deformation of a scheme Y at a closed point P given as $\mathrm{Spec} \mathcal{O}_{Y,P}/(\xi_1^n, \dots, \xi_e^n)$, where (ξ_1, \dots, ξ_e) is a system of parameters in $\mathcal{O}_{Y,P}$, with $X \subseteq \mathbb{A}_\kappa^{d+1}$ a hypersurface with equation $f = 0$, and with $X_{\eta_n} \subseteq \mathbb{A}_{\eta_n}^{d+1}$ the deformed hypersurface with equation $f - (\xi_1 \cdots \xi_e)^{n-1} = 0$, we set

$$\mathrm{Mil}_{X_{\mathrm{mot}}}^{(Y,P)}(t) := \sum_{n=1}^{\infty} \mathbb{L}^{-d\ell(\eta_n)} [\nabla_{j_{\eta_n}^*} X_{\eta_n}] t^n;$$

Poincare: with $\nabla_{\hat{Y}_P/j_n} X$ the submotif of $\nabla_{j_n} X$ consisting of all arcs that factor through the formal completion \hat{Y}_P , we set

$$\mathrm{Poin}_{X_{\mathrm{mot}}}^{(Y,P)}(t) := \sum_{n=1}^{\infty} \mathbb{L}^{-d\ell(j_n)} [\nabla_{\hat{Y}_P/j_n} X] t^n$$

Hasse-Weil: with $X^{(n)}$ the n -fold symmetric product of X , that is to say, the Hilbert scheme of effective zero cycles of degree n on X , we set

$$\mathrm{HW}_X^{\mathrm{mot}} := \sum_{n=0}^{\infty} [X^{(n)}] t^n.$$

Note that the specializations of the motivic Hilbert and Hilbert-Kunz series to the classical Grothendieck ring are trivial since the underlying varieties are just points. Put differently, this kind of motivic rationality cannot even be phrased in the classical setup. However, specializing to the length of the motives, which is well-defined by Proposition 14.4, yields their classical counterparts.

The final section, §15 is devoted to motivic integration. We only develop the finitistic theory, that is to say, over a fixed fat point, leaving the case of a limit point to a future paper. One of the great disadvantages of the categorical approach is that fibers are in general not functorial (after all, a fiber is the complement of the remaining fibers). We can overcome this by restricting to the category of *split fat points*, in which the morphisms are now assumed to be split. Our motivic integration will take values in the localization $\mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}$. A functor s , viewed on the category of split fat points, from a formal motif \mathfrak{X} on X to the constant sheaf with values in this localization $\mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}$ is called a *formal invariant* if all its fibers are formal motives, with only finitely many non-empty. We then define

$$\int s \, d_{\mathfrak{z}} X := \mathbb{L}^{-dl} \sum_{g \in \mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}} g \cdot [\nabla_{\mathfrak{z}}(s^{-1}(g))],$$

where d is the dimension of X and l the length of \mathfrak{z} . This motivic integral can be calculated locally (Theorem 15.8).

Notation and terminology. Varieties are assumed to be reduced, but not necessarily irreducible. Given a scheme X , we let X^{red} denote its *underlying variety or reduction*. We often denote a morphism of affine schemes $\text{Spec } B \rightarrow \text{Spec } A$ by the same letter as the corresponding ring homomorphism $A \rightarrow B$, whenever this causes no confusion. By a *germ* (X, Y) we mean a scheme X together with a closed subscheme $Y \subseteq X$. Most of the time Y is an irreducible subvariety, that is to say, the closure of a point $y \in X$, and we simply write (X, y) for this germ. If Y is a closed point, we call the germ *closed*. The n -th jet $J_Y^n X$ of a germ (X, Y) is the closed subscheme defined by \mathcal{I}_Y^n , where \mathcal{I}_Y is the ideal of definition of Y .¹ The *formal completion* \hat{X}_Y of the germ (X, Y) is the locally ringed space obtained as the direct limit of the $J_Y^n X$ (see [7, II.§9]). For instance, if $Y = P$ is a closed point with maximal ideal \mathfrak{m}_P , then the ring of global sections of \hat{X}_P is the \mathfrak{m}_P -adic completion $\hat{\mathcal{O}}_{X,P}$ of $\mathcal{O}_{X,P}$.

We often denote the affine line \mathbb{A}_κ^1 by L and the origin by O . The formal completion of the germ (L, O) is denoted \hat{L} , and the basic open $L - O$, that is to say, the open subscheme obtained by removing the origin, is denoted L_* . The respective classes, in whichever Grothendieck ring we consider, are denoted \mathbb{L} , $\hat{\mathbb{L}}$ and \mathbb{L}_* . Whereas in the classical Grothendieck ring, $\mathbb{L} = \mathbb{L}_* + 1$ (and $\hat{\mathbb{L}}$ is undefined), in the formal Grothendieck ring, we have $\mathbb{L} = \mathbb{L}_* + \hat{\mathbb{L}}$ (see Proposition 7.1), which, after taking global sections, takes the suggestive form

$$“\kappa[x]” = “\kappa[x, 1/x]” + “\kappa[[x]]”.$$

The n -th jet of (L, O) will be denoted $\mathfrak{l}_n := \text{Spec}(\kappa[x]/(x^n))$.

2. SIEVES

Fix a Noetherian scheme V , to be used as our base space, and which, for reasons that will become apparent soon, will also be assumed to be Jacobson. Most often, V is just the spectrum of an algebraically closed field κ . By a V -scheme X , we mean a separated scheme X together with a morphism of finite type $X \rightarrow V$. We call X a *fat V -point*, if $X \rightarrow V$ is finite and X has a unique point. In other words, $X = \text{Spec } R$, for some Artinian local ring R which is finite as a \mathcal{O}_V -module. We call the length of R the *length* of the fat point and denote it $\ell(\mathfrak{z})$. We denote the subcategory of fat V -points by \mathbf{Fat}_V , and we will use letters $\mathfrak{z}, \mathfrak{v}, \dots$ to denote fat points. An important example of fat points are the

¹Note that many authors take instead the $n + 1$ -th power.

jets of a closed germ (X, P) , that is to say, given a V -scheme X and a closed point P on X with corresponding maximal ideal \mathfrak{m}_P , we let $J_P^n X$, called the n -th jet of X along P , be the fat point with coordinate ring $\mathcal{O}_{X,P}/\mathfrak{m}_P^n$.

Let X be a V -scheme and \mathfrak{z} a fat point. A morphism of V -schemes $a: \mathfrak{z} \rightarrow X$ will be called a \mathfrak{z} -rational point on X (over V). The set of all \mathfrak{z} -rational points on X will be denoted by $X(\mathfrak{z})$. The image of the unique point of \mathfrak{z} under a is a closed point on X , called the *center* (or *origin*) of a . Indeed, since the composition $\mathfrak{z} \rightarrow X \rightarrow V$ is the structure map whence finite, so is its first component $a: \mathfrak{z} \rightarrow X$. As finite morphisms are proper, closed points are mapped to closed points. Let x be the center of a \mathfrak{z} -rational point a on a V -scheme X . We will denote the residue field of x and \mathfrak{z} respectively by $\kappa(x)$ and $\kappa(\mathfrak{z})$. The \mathfrak{z} -rational point a induces a homomorphism of residue fields $\kappa(x) \rightarrow \kappa(\mathfrak{z})$. By the Nullstellensatz this is a finite extension, and its degree will be called the *degree* of a . In particular, if V is the spectrum of an algebraically closed field κ , then $\kappa = \kappa(x) = \kappa(\mathfrak{z})$, so that any \mathfrak{z} -rational point has degree zero.

The category of contravariant functors (with natural transformations as morphisms) from $\mathbb{F}\text{at}_V$ to the category of sets will be called the category of *presheaves* over V , following standard practice in topos theory, notwithstanding the confusion this causes with the usual notion from algebraic geometry. The *product* $\mathfrak{X} \times \mathfrak{Y}$ (respectively, the *disjoint union* $\mathfrak{X} \sqcup \mathfrak{Y}$) of two presheaves \mathfrak{X} and \mathfrak{Y} is defined point-wise by the rule that a fat point \mathfrak{z} is mapped to the Cartesian product $\mathfrak{X}(\mathfrak{z}) \times \mathfrak{Y}(\mathfrak{z})$ (respectively, to the disjoint union $\mathfrak{X}(\mathfrak{z}) \sqcup \mathfrak{Y}(\mathfrak{z})$). Similarly, we say that \mathfrak{X} is a *sub-presheaf* of \mathfrak{Y} , symbolically $\mathfrak{X} \subseteq \mathfrak{Y}$, if for every fat point \mathfrak{z} , we have an inclusion $\mathfrak{X}(\mathfrak{z}) \subseteq \mathfrak{Y}(\mathfrak{z})$, and this inclusion is a natural transformation, meaning that for any morphism $j: \tilde{\mathfrak{z}} \rightarrow \mathfrak{z}$, we have a commutative diagram

$$(3) \quad \begin{array}{ccc} \mathfrak{X}(\mathfrak{z}) & \xrightarrow{\subseteq} & \mathfrak{Y}(\mathfrak{z}) \\ \mathfrak{X}(j) \downarrow & & \downarrow \mathfrak{Y}(j) \\ \mathfrak{X}(\tilde{\mathfrak{z}}) & \xrightarrow{\subseteq} & \mathfrak{Y}(\tilde{\mathfrak{z}}) \end{array}$$

where the downward arrows are the maps induced functorially by j . We call a presheaf \mathfrak{X} on $\mathbb{F}\text{at}_V$ *representable* (respectively, *pro-representable*), if there exists a V -scheme X (respectively, a scheme X which is not necessarily of finite type over V) such that $X(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z})$, for all fat points \mathfrak{z} . To emphasize that we view the (V) -scheme X as a contravariant functor on $\mathbb{F}\text{at}_V$, we will denote it by $X^\circ := \text{Mor}_V(\cdot, X)$. Any morphism $\varphi: Y \rightarrow X$ of (V) -schemes induces, by composition, a natural transformation $\varphi^\circ: Y^\circ \rightarrow X^\circ$, that is to say, a morphism in the category of presheaves on $\mathbb{F}\text{at}_V$. More precisely, given a fat point \mathfrak{z} and a \mathfrak{z} -rational point $b: \mathfrak{z} \rightarrow Y$, let $\varphi^\circ(\mathfrak{z})(b) := \varphi \circ b$. Instead of $\varphi^\circ(\mathfrak{z})$, we will simply write $\varphi(\mathfrak{z})$ for the induced map $Y(\mathfrak{z}) \rightarrow X(\mathfrak{z})$ if there is no danger for confusion.

2.1. Lemma. *Two closed subschemes X and Y of a V -scheme Z are distinct if and only if there is a fat point \mathfrak{z} such that $X(\mathfrak{z})$ and $Y(\mathfrak{z})$ are distinct subsets of $Z(\mathfrak{z})$.*

Proof. One direction is immediate, so assume X and Y are distinct. Then their restriction to some affine open of Z remains distinct, and hence we may assume $Z = \text{Spec } A$ is affine. Let I and J be the ideals in A defining X and Y respectively. Since $I \neq J$, there is a maximal ideal $\mathfrak{m} \subseteq A$ such that $IA_{\mathfrak{m}} \neq JA_{\mathfrak{m}}$. Hence, by Krull's Intersection Theorem in the Noetherian local ring $A_{\mathfrak{m}}$, there is some n such that I and J remain distinct ideals

in A/\mathfrak{m}^n . In particular, upon replacing I by J if necessary, $\mathfrak{z} := \operatorname{Spec}(A/(I + \mathfrak{m}^n))$ is a closed subscheme of X , but not of Y , showing that the closed immersion $\mathfrak{z} \subseteq Z$ lies in $X(\mathfrak{z}) - Y(\mathfrak{z})$. \square

The resulting map from the category of V -schemes to the category of presheaves on $\mathbb{F}\mathbf{at}_V$ is a full embedding by Yoneda's Lemma, Lemma 2.1, and Proposition 2.7 below. Note that this is no longer true for schemes not of finite type, an obvious reason for this failure being that there might be no rational points at all: for instance, $\operatorname{Spec}(\mathbb{C})$ has no rational points over any fat point defined over the algebraic closure $\bar{\mathbb{Q}}$. The product of two (pro-)representable presheaves is again (pro-)representable. More explicitly, the product $X^\circ \times Y^\circ$ is the same as $(X \times_V Y)^\circ$. If $s: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of presheaves, then we define its *image* $\operatorname{Im}(s)$ and its *graph* $\Gamma(s)$ as the sub-presheaf of respectively \mathfrak{X} and $\mathfrak{X} \times \mathfrak{Y}$, given at each fat point \mathfrak{z} as respectively the image and the graph of the map $s(\mathfrak{z}): \mathfrak{X}(\mathfrak{z}) \rightarrow \mathfrak{Y}(\mathfrak{z})$.

Sieves. By a *sieve*, we mean a sub-presheaf \mathfrak{X} of some representable X° . If we want to emphasize the underlying V -scheme X , we say that \mathfrak{X} is a *sieve on X* , or that X is an *ambient space* of \mathfrak{X} . Some examples of sieves: if $\varphi: Y \rightarrow X$ is a morphism of V -schemes, then we let $\operatorname{Im}(\varphi)$ be the image pre-sieve of the corresponding natural transformation $\varphi^\circ: Y^\circ \rightarrow X^\circ$, that is to say, $\operatorname{Im}(\varphi)(\mathfrak{z})$, for a fat point \mathfrak{z} , consists of all \mathfrak{z} -rational points on X that *lift* to a \mathfrak{z} -rational point on Y , meaning that $\mathfrak{z} \rightarrow X$ factors through Y . Any sieve of the form $\operatorname{Im}(\varphi)$ for some morphism $\varphi: Y \rightarrow X$ of V -schemes is called *sub-schemic*. If φ is a (locally) closed or open immersion, then $\operatorname{Im}(\varphi)$ is equal to Y° , whence is itself representable, and we call $\operatorname{Im}(\varphi) \cong Y^\circ$ respectively a *(locally) closed* or *open subsieve* on X .

2.2. Lemma. *Let \mathfrak{X} be a sieve on X and $\mathfrak{z} = \operatorname{Spec} R$ a fat point. A \mathfrak{z} -rational point $a: \mathfrak{z} \rightarrow X$ belongs to $\mathfrak{X}(\mathfrak{z})$ if and only if $\operatorname{Im}(a) \subseteq \mathfrak{X}$. If $\mathfrak{X} = Y^\circ$ is a closed subsieve, given by a closed subscheme $Y \subseteq X$, then this is also equivalent with $\mathfrak{z} \cong \mathfrak{z} \times_X Y$ and also with $a^* \mathcal{I}_Y = 0$, where \mathcal{I}_Y is the ideal sheaf of Y and $a^* \mathcal{I}_Y$ its image in R .*

Proof. Suppose $a \in \mathfrak{X}(\mathfrak{z})$. Let \mathfrak{w} be a fat point and $b \in \operatorname{Im}(a)(\mathfrak{w})$. Hence $b: \mathfrak{w} \rightarrow X$ factors through a , that is to say, we can find a morphism $i: \mathfrak{w} \rightarrow \mathfrak{z}$ such that the diagram

$$(4) \quad \begin{array}{ccc} & \mathfrak{w} & \\ i \swarrow & & \searrow b \\ \mathfrak{z} & \xrightarrow{a} & X \end{array}$$

commutes. By functoriality, i induces a map $\mathfrak{X}(\mathfrak{z}) \rightarrow \mathfrak{X}(\mathfrak{w})$, sending a to b , proving that $b \in \mathfrak{X}(\mathfrak{w})$. Since this holds for all \mathfrak{w} and b , we showed $\operatorname{Im}(a) \subseteq \mathfrak{X}$. Conversely, assume the latter inclusion of sieves holds. In particular, the identity $1_{\mathfrak{z}}$ is a \mathfrak{z} -rational point whose image under $a(\mathfrak{z})$ is just a , proving that $a \in \operatorname{Im}(a)(\mathfrak{z}) \subseteq \mathfrak{X}(\mathfrak{z})$.

To see the equivalence with the last two conditions if $\mathfrak{X} = Y^\circ$, we may work locally and assume $X = \operatorname{Spec} A$. Let I be the ideal defining Y , and let $A \rightarrow R$ be the homomorphism corresponding to a . Then $a \in Y(\mathfrak{z})$ if and only if $IR = 0$ if and only if $A/I \otimes_A R \cong R$, proving the desired equivalences. \square

We may generalize the notion of an image sieve as follows. Given a sieve \mathfrak{Y} on an V -scheme Y and a morphism $\varphi: Y \rightarrow X$ of V -schemes, we define the *push-forward* $\varphi_*\mathfrak{Y}$ as the sieve on X given at a fat point \mathfrak{z} as the image of $\mathfrak{Y}(\mathfrak{z}) \subseteq Y(\mathfrak{z})$ under the map $\varphi(\mathfrak{z}): Y(\mathfrak{z}) \rightarrow X(\mathfrak{z})$. In particular, $\varphi_*Y^\circ = \text{Im}(\varphi)$. Similarly, given a sieve \mathfrak{X} on X , we define its *pull-back* $\varphi^*\mathfrak{X}$ as the sieve on Y given at a fat point \mathfrak{z} as the pre-image of $\mathfrak{X}(\mathfrak{z})$ under the map $\varphi(\mathfrak{z}): Y(\mathfrak{z}) \rightarrow X(\mathfrak{z})$. In other words, $\varphi^*\mathfrak{X}(\mathfrak{z})$ consists of those rational points $\mathfrak{z} \rightarrow Y$ such that the composition $\mathfrak{z} \rightarrow Y \rightarrow X$ lies in $\mathfrak{X}(\mathfrak{z})$. The pull-back of a closed subsieve is again a closed subsieve: if $\bar{X} \subseteq X$ is a closed immersion and $\varphi: Y \rightarrow X$ a morphism, then $\varphi^*\bar{X}^\circ$ is the closed subsieve given by $\varphi^{-1}(\bar{X}) = Y \times_X \bar{X}$. More generally, if $\bar{X} \rightarrow X$ is an arbitrary morphism, then the pull-back of the sub-schemic sieve $\text{Im}(\bar{X} \rightarrow X)$ is the sub-schemic sieve $\text{Im}(Y \times_X \bar{X} \rightarrow Y)$ of the base change.

If \mathfrak{X} and \mathfrak{Y} are sieves on an V -scheme X , then we define their *intersection* $\mathfrak{X} \cap \mathfrak{Y}$ and *union* $\mathfrak{X} \cup \mathfrak{Y}$ as the sieves given respectively by point-wise intersection and union, that is to say, $(\mathfrak{X} \cap \mathfrak{Y})(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z}) \cap \mathfrak{Y}(\mathfrak{z}) \subseteq X(\mathfrak{z})$ and $(\mathfrak{X} \cup \mathfrak{Y})(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z}) \cup \mathfrak{Y}(\mathfrak{z}) \subseteq X(\mathfrak{z})$. One easily checks that they are again sieves, that is to say, (contravariant) functors. We express this by saying that the sieves on X form a *lattice*. Clearly, the intersection of two closed subsieves is again a closed subsieve: $V^\circ \cap W^\circ = (V \cap W)^\circ$, for $V, W \subseteq X$ closed subschemes, but this is no longer true for their union. The disjoint union $\mathfrak{X} \sqcup \mathfrak{Y}$ is equal to the union of the push-forwards of \mathfrak{X} and \mathfrak{Y} under the immersions $X \rightarrow X \sqcup Y$ and $Y \rightarrow X \sqcup Y$ respectively.

By the *Zariski closure* of \mathfrak{X} in X , denoted $\bar{\mathfrak{X}}$, we mean the intersection of all closed subschemes $Y \subseteq X$ such that $\mathfrak{X} \subseteq Y^\circ$. By Noetherianity, $\bar{\mathfrak{X}}$ is a sieve on its Zariski closure $\bar{\mathfrak{X}}$, and the latter is the smallest closed subscheme on which \mathfrak{X} is a sieve. We say that \mathfrak{X} is *Zariski dense* in X if $\bar{\mathfrak{X}} = X$. For instance, given a morphism $\varphi: Y \rightarrow X$ of V -schemes, the Zariski closure of $\text{Im}(\varphi)$ is the so-called *scheme-theoretic image* of φ , that is to say, the closed subscheme of X given by the kernel of the induced morphism $\mathcal{O}_X \rightarrow \varphi_*\mathcal{O}_Y$. When $\text{Im}(\varphi)$ is Zariski dense, one says that φ is *dominant*.

2.3. Lemma. *If $\mathfrak{w} \rightarrow \mathfrak{v}$ is a dominant morphism of fat points, then the induced map $\mathfrak{X}(\mathfrak{v}) \rightarrow \mathfrak{X}(\mathfrak{w})$ is injective, for any sieve \mathfrak{X} .*

Proof. This is a form of duality: dominant morphisms are epimorphisms and so under a contravariant functor they become monomorphisms. More explicitly, since $\mathfrak{X} \subseteq X^\circ$, for some V -scheme X , it suffices to show injectivity for the latter, that is to say, we may assume \mathfrak{X} is representable, and then since the problem is local, we may assume X is affine with coordinate ring A . Let $a, a' \in X(\mathfrak{v})$ have the same image in $X(\mathfrak{w})$. If $R \rightarrow S$ is the homomorphism corresponding to φ , then dominance means that this homomorphism is injective. Since, by assumption, the two homomorphisms $A \rightarrow R$ induced respectively by a and a' give rise to the same homomorphism $A \rightarrow S$ when composed with the injection $R \subseteq S$, they must already be equal, as we needed to show. \square

2.4. Example. Suppose V is the spectrum of an algebraically closed field κ . Zariski closure does not commute with taking κ -rational points, that is to say, the κ -rational points of the Zariski closure of a sieve \mathfrak{X} in X may be bigger than the Zariski closure of $\mathfrak{X}(\kappa)$ (in the usual Zariski topology) in $X(\kappa)$. For instance, as we will see shortly, the cone $\mathfrak{C}(O)$ of the origin O on the affine line \mathbb{A}_κ^1 has Zariski closure equal to \mathbb{A}_κ^1 , whereas its κ -rational points consist just of the origin.

Germes of sieves. Strictly speaking, a sieve is a pair (\mathfrak{X}, X) consisting of a sub-presheaf \mathfrak{X} of an ambient space X , but we will often treat sieves as abstract objects, that is to say, disregarding their ambient space. This allows us, for instance, to view the same sieve as

already defined on a smaller subscheme. To give a more formal treatment, let us call a *germ of a sieve* any equivalence class of pairs (\mathfrak{Y}, Y) , where we call two such pairs (\mathfrak{Y}, Y) and (\mathfrak{Y}', Y') equivalent if there exists a pair (\mathfrak{X}, X) and locally closed immersions $\varphi: Y \rightarrow X$ and $\varphi': Y' \rightarrow X'$ such that $\varphi_* \mathfrak{Y} = \mathfrak{X} = \varphi'_* \mathfrak{Y}'$. In particular, \mathfrak{Y} is then also a sieve on the intersection $Y \cap Y'$, viewed as a locally closed subscheme of X . Therefore, we may always assume, if necessary, that \mathfrak{Y} is Zariski dense in Y , and then it is also a Zariski dense sieve on any open subsieve of Y containing it. Henceforth, we will often confuse a sieve with the germ it determines.

Complete sieves. The complement of a sieve $\mathfrak{X} \subseteq X^\circ$ is in general not a sieve, as witnessed by any closed subsieve. Let us call a sieve \mathfrak{X} on X *complete* if

$$\mathfrak{X}(\mathfrak{z}) = X(j)^{-1}(\mathfrak{X}(\tilde{\mathfrak{z}})),$$

for every morphism $j: \tilde{\mathfrak{z}} \rightarrow \mathfrak{z}$ of fat points, where $X(j): X(\tilde{\mathfrak{z}}) \rightarrow X(\mathfrak{z})$ is the map induced functorially by j . More generally, if $\mathfrak{Y} \subseteq \mathfrak{X}$ is an inclusion of sieves on X , then we say that \mathfrak{Y} is *relatively complete in \mathfrak{X}* , if

$$\mathfrak{Y}(\mathfrak{z}) = X(j)^{-1}(\mathfrak{Y}(\tilde{\mathfrak{z}})) \cap \mathfrak{X}(\mathfrak{z}),$$

for all morphisms j .

If V is the spectrum of an algebraically closed field κ , then V itself is a fat point, and any fat point \mathfrak{z} admits a unique morphism $V \rightarrow \mathfrak{z}$ (given by the residue field of \mathfrak{z}). We let $\rho_{\mathfrak{z}}: X(\mathfrak{z}) \rightarrow X(\kappa)$ be the induced map. One easily verifies that \mathfrak{Y} is relatively complete in \mathfrak{X} if and only if $\mathfrak{Y}(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z}) \cap \rho_{\mathfrak{z}}^{-1}(\mathfrak{Y}(\kappa))$, for every fat point \mathfrak{z} . In fact, for V arbitrary, we have a similar criterion in that we only have to check the condition for every morphisms of fat points $j: \tilde{\mathfrak{z}} \rightarrow \mathfrak{z}$ in which $\tilde{\mathfrak{z}}$ has length one (necessarily therefore the spectrum of a field extension of the residue field of \mathfrak{z}).

As we shall see in Proposition 7.1 below, open subsieves over an algebraically closed field are complete, and their complements are again sieves. The second property in fact follows from the first, as one easily verifies:

2.5. Lemma. *Given an inclusion of sieves $\mathfrak{Y} \subseteq \mathfrak{X}$, then \mathfrak{Y} is relatively complete in \mathfrak{X} if and only if $\mathfrak{X} - \mathfrak{Y}$ is again a sieve.* \square

Assume V is the spectrum of an algebraically closed field κ . Given a κ -scheme X and a subset $V \subseteq X(\kappa)$, we define the *cone $\mathfrak{C}(V)$ over V* to be the sieve given by

$$\mathfrak{C}(V)(\mathfrak{z}) := \rho_{\mathfrak{z}}^{-1}(V)$$

for every fat point \mathfrak{z} . By our previous discussion on complete sieves over an algebraically closed field, it follows that $\mathfrak{C}(V)$ is complete, and its complement is the cone $\mathfrak{C}(-V)$. More generally, for any sieve \mathfrak{X} , the intersection $\mathfrak{X} \cap \mathfrak{C}(V)$ is relatively complete in \mathfrak{X} , and its complement in \mathfrak{X} is equal to $\mathfrak{X} \cap \mathfrak{C}(-V)$. We have the following converse:

2.6. Lemma. *Over an algebraically closed field κ , a subsieve $\mathfrak{Y} \subseteq \mathfrak{X}$ is relatively complete if and only if it is the intersection of \mathfrak{X} and a cone.*

Proof. Let \mathfrak{Y} be relatively complete in \mathfrak{X} , and let $V := \mathfrak{Y}(\kappa)$. Then $\mathfrak{Y} = \mathfrak{C}(V) \cap \mathfrak{X}$, since both have the same \mathfrak{z} -rational points, for any fat point \mathfrak{z} . \square

Given a sieve \mathfrak{X} , we define its *completion* as the cone $\hat{\mathfrak{X}} := \mathfrak{C}(\mathfrak{X}(\kappa))$. By functoriality, \mathfrak{X} is contained in $\hat{\mathfrak{X}}$, and $\hat{\mathfrak{X}}$ is the smallest complete sieve containing \mathfrak{X} .

The category of sieves. By definition, a *morphism* between sieves is a natural transformation, giving rise to the *category of V -sieves*, denoted \mathbf{Sieve}_V . If V is Jacobson,² then the category of V -schemes fully embeds in the category \mathbf{Sieve}_V , by the following result:

2.7. Proposition. *Assume V is Jacobson. Let \mathfrak{X} be a sieve on a V -scheme X , and let Y be a V -scheme. Any morphism $s: Y^\circ \rightarrow \mathfrak{X}$ is induced by a morphism $\varphi: Y \rightarrow X$, that is to say, $s = \varphi^\circ$ and $\text{Im}(\varphi) \subseteq \mathfrak{X}$.*

Proof. Assume first that $Y = \mathfrak{z}$ is a fat point, and $\mathfrak{X} = X^\circ$ is representable. Let $q := s(\mathfrak{z})(1_{\mathfrak{z}})$, where $1_{\mathfrak{z}}$ is the identity morphism on \mathfrak{z} viewed as a \mathfrak{z} -rational point on \mathfrak{z} . Hence $q \in X(\mathfrak{z})$, that is to say, a V -morphism $\mathfrak{z} \rightarrow X$. We have to show that $s = q^\circ$, so let \mathfrak{w} be any fat point, and $a: \mathfrak{w} \rightarrow \mathfrak{z}$ a \mathfrak{w} -rational point on \mathfrak{z} . By definition of q° , we have $q(\mathfrak{w})(a) = qa$. On the other hand, functoriality yields a commutative diagram

$$(5) \quad \begin{array}{ccc} \mathfrak{z}(\mathfrak{z}) & \xrightarrow{\mathfrak{z}^\circ(a)} & \mathfrak{z}(\mathfrak{w}) \\ s(\mathfrak{z}) \downarrow & & \downarrow s(\mathfrak{w}) \\ X(\mathfrak{z}) & \xrightarrow{X^\circ(a)} & X(\mathfrak{w}) \end{array}$$

We trace the image of $1_{\mathfrak{z}}$ in $X(\mathfrak{w})$ through this diagram. The top arrow sends it to a and hence its image in $X(\mathfrak{w})$ is $s(\mathfrak{w})(a)$. On the other hand, $s(\mathfrak{z})(1_{\mathfrak{z}}) = q$ and its image under $X^\circ(a)$ is qa , showing that $s(\mathfrak{w})(a) = qa = q(\mathfrak{w})(a)$, whence $s = q^\circ$.

Assume next that Y is arbitrary. Let $Z \subseteq Y$ be a closed subscheme of dimension zero. There exist finitely many fat points $\mathfrak{z}_1, \dots, \mathfrak{z}_s \subseteq Y$ such that $Z = \mathfrak{z}_1 \sqcup \dots \sqcup \mathfrak{z}_s$. By what we just proved, there exists a morphism $q_Z: Z \rightarrow X$ such that $s|_{Z^\circ} = q_Z^\circ$, for each zero-dimensional closed subscheme $Z \subseteq Y$. By Lemma 2.8 below, the direct limit of all such closed subschemes Z is Y , and hence, by the universal property of direct limits, we get a morphism $\varphi: Y \rightarrow X$, such that $q_Z = \varphi|_Z$. Checking at every fat point, we see that $\varphi^\circ = s$. Finally, assume \mathfrak{X} is an arbitrary sieve. By what we just proved, the composition $Y^\circ \rightarrow \mathfrak{X} \subseteq X^\circ$ is induced by a morphism $\varphi: Y \rightarrow X$. Since the image of $s = \varphi^\circ$ at any fat point \mathfrak{z} lies inside $\mathfrak{X}(\mathfrak{z})$, we must have $\text{Im}(\varphi) \subseteq \mathfrak{X}$. \square

2.8. Lemma. *If V is Jacobson, then any V -scheme is the direct limit of its zero-dimensional closed subschemes.*

Proof. I will only consider the case that V is the spectrum of a field κ . Reasoning on a finite open affine covering, we may further assume that $X = \text{Spec } A$ is an affine κ -scheme. Let \tilde{A} be the inverse limit of all residue rings A/\mathfrak{n} of finite length (we say that \mathfrak{n} has *finite colength*). We have to show that $A = \tilde{A}$. The inclusion $A \subseteq \tilde{A}$ follows from the fact that the intersection of all ideals of finite colength is zero.

Before we give the proof, we establish a preservation result under finite maps: if $A = \tilde{A}$ and $A \rightarrow B$ is a finite homomorphism, then also $B = \tilde{B}$. Indeed, if β_1, \dots, β_s generate B as an A -module, then they also generate every $B/\mathfrak{n}B$ as an A/\mathfrak{n} -module, where A/\mathfrak{n} has finite length. Since the $\mathfrak{n}B$ are cofinal in the set of all ideals of B of finite colength, \tilde{B} is generated by the β_i as an \tilde{A} -module, and hence $B = \tilde{B}$. This argument also shows that

²A scheme is called *Jacobson* if it admits an open covering by affine Jacobson schemes; an affine scheme $\text{Spec } A$ is Jacobson, if A is, meaning that any radical ideal is the intersection of the maximal ideals containing it.

$(\widetilde{A/I})$ is equal to $\tilde{A}/I\tilde{A}$, for any ideal $I \subseteq A$. By Noether Normalization, A is a finite extension of a polynomial ring over κ in the same number of variables as the dimension d of A . Hence, it will suffice to show the identity $A = \tilde{A}$ for polynomial rings.

We induct on d , where the case $d = 0$ is trivial. Let $A = \kappa[x]$ with x a d -tuple of variables, and let \mathfrak{o} be a complete discrete valuation ring containing A and having non-zero center $\mathfrak{p} \subseteq A$ (recall that \mathfrak{p} is the prime ideal of elements in A of positive value). Assume first that $d = 1$, so that \mathfrak{p} is a maximal ideal. Since \tilde{A} is then contained in the completion $\hat{A}_{\mathfrak{p}}$ (as the latter is the inverse limit of all A/\mathfrak{p}^n), and since by the universal property of completion $\hat{A}_{\mathfrak{p}} \subseteq \mathfrak{o}$, we showed $\tilde{A} \subseteq \mathfrak{o}$. Since the intersection of all these complete discrete valuation rings is equal to the normalization of A , whence to A , we showed $A = \tilde{A}$ in this case. As mentioned above, Noether Normalization then proves the result for all one-dimensional algebras, and so we may assume $d > 1$. If \mathfrak{p} is a maximal ideal, the previous argument yields again $\tilde{A} \subseteq \mathfrak{o}$. So assume \mathfrak{p} is a non-maximal, non-zero prime ideal. Since A/\mathfrak{p}^n has dimension less than d , induction plus preservation under homomorphic images then yields $A/\mathfrak{p}^n = \tilde{A}/\mathfrak{p}^n\tilde{A}$. In particular, the \mathfrak{p} -adic completion of \tilde{A} is equal to $\hat{A}_{\mathfrak{p}}$, and hence is contained in \mathfrak{o} . Hence we showed that $\tilde{A} \subseteq \mathfrak{o}$, for any such discrete valuation ring \mathfrak{o} , so that the same argument as above yields $A = \tilde{A}$. \square

2.9. Corollary. *Suppose V is Jacobson. For any sieve \mathfrak{X} , we have an isomorphism of sieves*

$$\mathrm{Mor}_{\mathrm{Sieve}_V}((\cdot)^\circ, \mathfrak{X}) \cong \mathfrak{X}(\cdot).$$

Proof. Let \mathfrak{z} be a fat point and $s: \mathfrak{z}^\circ \rightarrow \mathfrak{X}$ a morphism of sieves. By Proposition 2.7, this morphism is induced by a morphism $a: \mathfrak{z} \rightarrow X$ of schemes, where X is some ambient space of \mathfrak{X} . Since $s = a^\circ$, we have an inclusion $\mathrm{Im}(a) \subseteq \mathfrak{X}$, and hence $a \in \mathfrak{X}(\mathfrak{z})$ by Lemma 2.2. The converse follows along the same lines. \square

Inspired by the result in Proposition 2.7, we call a natural transformation $s: \mathfrak{Y} \rightarrow \mathfrak{X}$ between sieves *algebraic*, if there exists a morphism of V -schemes $\varphi: Y \rightarrow X$ such that \mathfrak{X} and \mathfrak{Y} are sieves on respectively X and Y , and such that s is the restriction of $\varphi^\circ: Y^\circ \rightarrow X^\circ$; we might also express this by saying that s *extends* to a morphism of schemes. Since the definition allows the ambient spaces to be dependent on s , we have actually defined a morphism between germs of sieves. It follows that if \mathfrak{Y} is Zariski dense in Y and \mathfrak{X} is a sieve on X (without any further restriction), then s extends to a morphism $\tilde{Y} \rightarrow X$ for some open $\tilde{Y} \subseteq Y$ on which \mathfrak{Y} is also a sieve, that is to say, such that $\mathfrak{Y} \subseteq \tilde{Y}^\circ$.

The composition of two algebraic natural transformations is again algebraic. Indeed, let $s: \mathfrak{Z} \rightarrow \mathfrak{Y}$ and $t: \mathfrak{Y} \rightarrow \mathfrak{X}$ be algebraic, extending respectively to morphisms $\varphi: Z \rightarrow Y$ and $\psi: Y' \rightarrow X$. Since Y and Y' are both ambient spaces for \mathfrak{Y} , so is their intersection $Y'' := Y \cap Y'$, which is therefore locally closed in either. Hence the restriction of φ to $\varphi^{-1}(Y'')$ (respectively, the restriction of ψ to Y'') is a morphism extending s (respectively t), and therefore, the composition $\varphi^{-1}(Y'') \rightarrow X$ extends $t \circ s$. We can therefore define the *explicit category of sieves*, denoted Sieve_V , as the subcategory of all sieves in which the morphisms are only the algebraic ones.

A note of caution: not every morphism of sieves is algebraic, and we will discuss some examples later. Moreover, even if it is, one cannot always extend it to any ambient space of the source sieve. An example is in order:

2.10. Example. Let $H \subseteq \mathbb{A}_\kappa^2$ be the hyperbola with equation $xy = 1$ over a field κ , and let L_* be the *punctured line*, that is to say, the affine line minus the origin. Note that this is an affine scheme with coordinate ring $\kappa[x, 1/x]$. The projection $\mathbb{A}_\kappa^2 \rightarrow \mathbb{A}_\kappa^1$ onto

the first coordinate induces an isomorphism $H \rightarrow L_*$. Its inverse induces an isomorphism $L_*^\circ \rightarrow H^\circ$, which is trivially algebraic, as both sieves are representable. However, although L_*° is an open subsieve on \mathbb{A}_κ^1 , the above isomorphism does not extend (since $1/x$ is not a polynomial).

We may generalize the definitions of pull-back and push-forward along a morphism of sieves as follows. Let $s: \mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism of sieves. Given a subsieve $\mathfrak{Y}' \subseteq \mathfrak{Y}$, we define its *push-forward* $s_* \mathfrak{Y}'$ as the presheaf defined at each fat point \mathfrak{z} as the image of $\mathfrak{Y}'(\mathfrak{z})$ under $s(\mathfrak{z})$. Similarly, given a subsieve $\mathfrak{X}' \subseteq \mathfrak{X}$, we define its *pull-back* $s^* \mathfrak{X}'$ as the presheaf defined at each fat point \mathfrak{z} as the pre-image of $\mathfrak{X}'(\mathfrak{z})$ under $s(\mathfrak{z})$.

3. THE TOPOS OF SIEVES

In view of Proposition 2.7, we will henceforth assume that the base space V is Jacobson. Recall that $\mathbb{A}_V^n := \mathbb{A}_{\mathbb{Z}}^n \times V$ is the *affine n -space* over V .

Section rings. Given a sieve \mathfrak{X} on X , we define its *global section ring* as

$$H_0(\mathfrak{X}) := \text{Mor}(\mathfrak{X}, \mathbb{A}_V^1),$$

where by the latter, we actually mean the collection of morphisms $\mathfrak{X} \rightarrow (\mathbb{A}_V^1)^\circ$, but for notational simplicity, we will identify a scheme with the functor it represents if there is no danger for confusion. For each fat point $\mathfrak{z} = \text{Spec } R$, we have a natural bijection $\Psi_{\mathfrak{z}}: \mathbb{A}_V^1(\mathfrak{z}) \cong R$ defined as follows. Given a rational point $a: \mathfrak{z} \rightarrow \mathbb{A}_V^1$, it factors through an affine open of the form $\mathbb{A}_\lambda^1 \subseteq \mathbb{A}_V^1$, for some affine open $\text{Spec } \lambda \subseteq V$, and hence induces a homomorphism $\lambda[y] \rightarrow R$, which, again for notational simplicity, we denote again by a . Now, set $\Psi_{\mathfrak{z}}(a) := a(y) \in R$. This identification endows $\mathbb{A}_V^1(\mathfrak{z})$ with a ring structure, and by transfer, then makes $H_0(\mathfrak{X})$ into a ring. Indeed, given sub-schematic morphisms $s, t: \mathfrak{X} \rightarrow \mathbb{A}_V^1$, we define their sum $s + t$ (respectively, their product st) as the morphism which at a fat point \mathfrak{z} maps $a \in \mathfrak{X}(\mathfrak{z})$ to

$$s(\mathfrak{z})(a) + t(\mathfrak{z})(a) = \Psi_{\mathfrak{z}}^{-1}(\Psi_{\mathfrak{z}}(s(\mathfrak{z})(a)) + \Psi_{\mathfrak{z}}(t(\mathfrak{z})(a)))$$

and a similar formula for $s(\mathfrak{z})(a) \cdot t(\mathfrak{z})(a)$. The functoriality of $s + t$ and st is easy, showing that they are again global sections.

As we shall see below, algebraic morphisms will play a key role, and so we define the ring of *algebraic sections* $H_0^{\text{geom}}(\mathfrak{X})$ as the subset of $H_0(\mathfrak{X})$ consisting of all algebraic morphisms $\mathfrak{X} \rightarrow \mathbb{A}_V^1$. To see that this is closed under sums and products, let s and s' be two algebraic sections. By definition, there exist morphisms $X \rightarrow \mathbb{A}_V^1$ and $X' \rightarrow \mathbb{A}_V^1$ inducing s and s' respectively, where X and X' are ambient spaces for \mathfrak{X} . In particular, the locally closed subscheme $X'' := X \cap X'$ is then also an ambient space for \mathfrak{X} . Hence $s + t$ and st extend to the sum and product of the restrictions to X'' of $X \rightarrow \mathbb{A}_V^1$ and $X' \rightarrow \mathbb{A}_V^1$, proving that they are also algebraic. By Theorem 3.7, every global section on a schematic motif is algebraic.

3.1. Lemma. *The assignment $\mathfrak{X} \mapsto H_0(\mathfrak{X})$ is a contravariant functor from \mathbf{Sieve}_V to the category of \mathcal{O}_V -algebras. Similarly, $\mathfrak{X} \mapsto H_0^{\text{geom}}(\mathfrak{X})$ is a contravariant functor on the explicit category \mathbf{Sieve}_V . In particular, if two sieves are isomorphic, then they have the same global section ring, and if they are algebraically isomorphic, then they have the same algebraic section ring.*

Proof. If $s: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of sieves, then pulling-back induces a \mathcal{O}_V -algebra homomorphism $H_0(\mathfrak{Y}) \rightarrow H_0(\mathfrak{X})$ given by $t \mapsto t \circ s$, for $t: \mathfrak{Y} \rightarrow \mathbb{A}_V^1$. One easily verifies that this constitutes a contravariant functor. Since the pull-back of a algebraic

morphism under an algebraic morphism is easily seen to be algebraic again, we get an induced homomorphism $H_0^{\text{geom}}(\mathfrak{Y}) \rightarrow H_0^{\text{geom}}(\mathfrak{X})$. \square

3.2. Proposition. *The global section ring of a representable functor X° is equal to the ring of global sections $H_0(X) := \Gamma(\mathcal{O}_X, X)$ of the corresponding scheme, and this is also its algebraic section ring.*

Proof. A morphism $X^\circ \rightarrow \mathbb{A}_V^1$ corresponds by Proposition 2.7 to a morphism of V -schemes $X \rightarrow \mathbb{A}_V^1$, and it is well-known that the collection of all these is precisely the ring of global sections on X (see, for instance, [7, II. Exercise 2.4]). \square

In particular, if \mathfrak{X} is a sieve on an affine scheme $\text{Spec } A$, then $H_0(\mathfrak{X})$ and $H_0^{\text{geom}}(\mathfrak{X})$ are A -algebras.

3.3. Corollary. *The algebraic section ring $H_0^{\text{geom}}(\mathfrak{X})$ of a sieve \mathfrak{X} is the inverse limit of all $H_0(X)$, where X runs over all ambient spaces of \mathfrak{X} .*

Proof. If $s: \mathfrak{X} \rightarrow \mathbb{A}_V^1$ is an algebraic section, then it extends to a morphism $X \rightarrow \mathbb{A}_V^1$, where X is some ambient space of \mathfrak{X} , and hence by the argument in the above proof, it is the image of an element in $H_0(X)$ under the homomorphism $H_0(X) \rightarrow H_0^{\text{geom}}(\mathfrak{X})$ induced by the inclusion $\mathfrak{X} \subseteq X^\circ$. If $X' \subseteq X$ is a locally closed subscheme which is also an ambient space for \mathfrak{X} , then s extends to a morphism with domain X' , and this is therefore necessarily the restriction of the global section in $H_0(X)$ determined by s . This shows that the $H_0(X)$ form an inverse system as X varies over the germ of \mathfrak{X} , with limit equal to $H_0^{\text{geom}}(\mathfrak{X})$. \square

3.4. Example. If \mathfrak{X} is Zariski dense in X , then the only ambient spaces of \mathfrak{X} inside X are open, so that we may think of the ring of algebraic sections as a sort of stalk:

$$H_0^{\text{geom}}(\mathfrak{X}) = \varprojlim H_0^{\text{geom}}(U)$$

where U runs over all open subschemes on which \mathfrak{X} is a sieve. In particular, if V is the spectrum of an algebraically closed field κ and $\mathfrak{X}(\kappa) = X(\kappa)$, then there are no proper opens in X on which \mathfrak{X} is a sieve, and hence $H_0^{\text{geom}}(\mathfrak{X}) = H_0(X)$. This holds automatically if X is for instance a fat point.

3.5. Theorem. *If \mathfrak{X} and \mathfrak{Y} are sieves on a common V -scheme, then the natural commutative diagram*

$$(6) \quad \begin{array}{ccc} H_0(\mathfrak{X} \cup \mathfrak{Y}) & \xrightarrow{i_{\mathfrak{Y}}} & H_0(\mathfrak{Y}) \\ \downarrow i_{\mathfrak{X}} & & \downarrow p_{\mathfrak{Y}} \\ H_0(\mathfrak{X}) & \xrightarrow{p_{\mathfrak{X}}} & H_0(\mathfrak{X} \cap \mathfrak{Y}) \end{array}$$

is Cartesian. In particular, if \mathfrak{X} and \mathfrak{Y} are disjoint, then $H_0(\mathfrak{X} \cup \mathfrak{Y}) \cong H_0(\mathfrak{X}) \oplus H_0(\mathfrak{Y})$. Similar properties hold for the algebraic section ring.

Proof. The construction of the commutative square (6) and the verification that it is commutative, follows easily from Lemma 3.1. Recall that (6), as a commutative square in the category of \mathcal{O}_V -algebras, is called *Cartesian* or a *pull-back*, if $H_0(\mathfrak{X} \cup \mathfrak{Y})$ is universal in this category for making the diagram commute, or, equivalently, if it is the equalizer of the

two compositions $p_{\mathfrak{X}}i_{\mathfrak{X}}$ and $p_{\mathfrak{Y}}i_{\mathfrak{Y}}$. To verify this property, let $s \in H_0(\mathfrak{X})$ and $t \in H_0(\mathfrak{Y})$ be such that $s|_{\mathfrak{X} \cap \mathfrak{Y}} = p_{\mathfrak{X}}(s) = p_{\mathfrak{Y}}(t) = t|_{\mathfrak{X} \cap \mathfrak{Y}}$. We then define $u \in H_0(\mathfrak{X} \cup \mathfrak{Y})$ as follows. Given a fat point \mathfrak{z} , let $u(\mathfrak{z})$ be the map sending $a \in \mathfrak{X}(\mathfrak{z}) \cup \mathfrak{Y}(\mathfrak{z})$ to $s(\mathfrak{z})(a)$ if $a \in \mathfrak{X}(\mathfrak{z})$, and to $t(\mathfrak{z})(a)$ if $a \in \mathfrak{Y}(\mathfrak{z})$. This is well-defined, since $s(\mathfrak{z})$ and $t(\mathfrak{z})$ agree on $\mathfrak{X}(\mathfrak{z}) \cap \mathfrak{Y}(\mathfrak{z})$ by assumption. It is now easy to verify that this defines a morphism of sieves $u: \mathfrak{X} \cup \mathfrak{Y} \rightarrow \mathbb{A}_V^1$, that is to say, a global section of $\mathfrak{X} \cup \mathfrak{Y}$, and that $i_{\mathfrak{X}}(u) = s$ and $i_{\mathfrak{Y}}(u) = t$. Moreover, if s and t are algebraic, then so is u . \square

3.6. Remark. To formulate a version for more than two sieves, we resort to the language of sheaf theory (and, shortly, we will put everything in this context anyway): given a union $\mathfrak{X} = \mathfrak{X}_1 \cup \dots \cup \mathfrak{X}_s$, we have an exact sequence

$$(7) \quad 0 \rightarrow H_0(\mathfrak{X}) \rightarrow \bigoplus_i H_0(\mathfrak{X}_i) \xrightarrow[\delta_{ji}]{\delta_{ij}} \bigoplus_{i < j} H_0(\mathfrak{X}_i \cap \mathfrak{X}_j)$$

where $\delta_{ij}: H_0(\mathfrak{X}_i) \rightarrow H_0(\mathfrak{X}_i \cap \mathfrak{X}_j)$ is the restriction homomorphism on the global sections induced by the inclusion $\mathfrak{X}_i \cap \mathfrak{X}_j \subseteq \mathfrak{X}_i$. The corresponding exact sequence for algebraic sections is

$$(8) \quad 0 \rightarrow H_0^{\text{geom}}(\mathfrak{X}) \rightarrow \bigoplus_i H_0^{\text{geom}}(\mathfrak{X}_i) \xrightarrow[\delta_{ji}]{\delta_{ij}} \bigoplus_{i < j} H_0^{\text{geom}}(\mathfrak{X}_i \cap \mathfrak{X}_j).$$

3.7. Theorem. *Any morphism $s: \mathfrak{X} \rightarrow \mathfrak{Z}$ of sieves with \mathfrak{X} sub-schematic, is algebraic. In particular, $H_0(\mathfrak{X}) = H_0^{\text{geom}}(\mathfrak{X})$.*

Proof. Let us prove the second assertion first. Let $\varphi: Y \rightarrow X$ be a morphism of V -schemes, so that $\mathfrak{X} = \text{Im}(\varphi)$. Replacing X by the Zariski closure of $\text{Im}(\varphi)$, we may assume that φ is dominant. Our objective is to show that we have an equality

$$(9) \quad H_0^{\text{geom}}(\text{Im}(\varphi)) = H_0(\text{Im}(\varphi)).$$

Assume first that $Y = \eta$ is a fat point, and hence, since φ is dominant, so is then $X = \mathfrak{x}$. Consider the induced homomorphism of Artinian local rings $R \subseteq S$, which is injective precisely because φ is dominant. Let $s: \text{Im}(\varphi) \rightarrow \mathbb{A}_V^1$ be a global section. By Proposition 2.7, we can find $q \in S$ which, when viewed as a global section $\eta \rightarrow \mathbb{A}_V^1$, extends the composition $s \circ \varphi^\circ$. Let $g_1: S \rightarrow S \otimes_R S$ and $g_2: S \rightarrow S \otimes_R S$ be given by respectively $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$. Since g_1 and g_2 agree on R , the two corresponding rational points $\eta \times_{\mathfrak{x}} \eta \rightarrow \eta$ have the same image under $s(\eta \times_{\mathfrak{x}} \eta)$ and hence, $g_1(q) = g_2(q)$. By Lemma 3.10 below, we get $q \in R$.

By Corollary 3.3, this proves (9) whenever the domain of φ is a fat point. If the domain is a zero-dimensional scheme Z , then it is a finite disjoint union $\eta_1 \sqcup \dots \sqcup \eta_s$ of fat points. By an induction argument, we may assume $s = 2$, so that $\text{Im}(\varphi) = \text{Im}(\varphi|_{\eta_1}) \cup \text{Im}(\varphi|_{\eta_2})$. In particular, $H_0(\text{Im}(\varphi))$ and $H_0^{\text{geom}}(\text{Im}(\varphi))$ satisfy each the same Cartesian square (6) by what we just proved for fat points (instead of induction, we may alternatively use (7) and (8)). By uniqueness, they must therefore be equal, showing that (9) holds whenever the domain is zero-dimensional. For Y arbitrary, we may write it as the direct limit of all its zero-dimensional closed subschemes by Lemma 2.8. Let $\{\mathfrak{Z} := \text{Im}(\varphi|_Z)\}$ be the collection of all sub-schematic motives, where $Z \subseteq Y$ varies over all zero-dimensional closed subschemes of Y . Since the \mathfrak{Z} form a direct system, $H_0(\text{Im}(\varphi))$ is the inverse limit of all $H_0(\mathfrak{Z}) = H_0^{\text{geom}}(\mathfrak{Z})$, by Lemma 3.9 below, and by the same argument, this direct limit is equal to $H_0^{\text{geom}}(\text{Im}(\varphi))$, proving (9).

To prove the first assertion, we may assume, without loss of generality, that $\mathfrak{Z} = Z^\circ$ is representable, and then, since the problem is local, that $X = \text{Spec } A$ and Z are affine.

Hence $Z \subseteq \mathbb{A}_\lambda^n$ is a closed subscheme for some n and some open $\text{Spec } \lambda \subseteq V$. Let s_i be the composition of s with the morphism induced by the projection $\mathbb{A}_\lambda^n \rightarrow \mathbb{A}_\lambda^1$ onto the i -th coordinate. Hence $s_i \in H_0(\mathfrak{X}) = H_0^{\text{geom}}(\mathfrak{X})$. Replacing X by an open subscheme if necessary as per Corollary 3.3, we can find $q_i \in A$ such that, viewed as a global section $X \rightarrow \mathbb{A}_\lambda^1$, it extends s_i . Therefore, the morphism $X \rightarrow \mathbb{A}_\lambda^n$ given by (q_1, \dots, q_n) is an extension of s , as we wanted to show. \square

3.8. Remark. By the above proof, we actually showed that if the domain of φ is zero-dimensional, then $H_0(\text{Im}(\varphi))$ is equal to the global section ring of the Zariski closure of $\text{Im}(\varphi)$. However, this is no longer true in the general case, as can be seen from Corollary 3.3.

3.9. Lemma. *Let $\{\mathfrak{Z}\}$ be a direct system of sieves on some scheme X and let \mathfrak{X} be their direct limit. Then $H_0(\mathfrak{X})$ is the inverse limit of all $H_0(\mathfrak{Z})$.*

Proof. Contravariance turns a direct limit into an inverse limit, and the rest is now an easy consequence of the universal property of inverse limits:

$$\begin{aligned} H_0(\mathfrak{X}) &= \text{Mor}(\mathfrak{X}, \mathbb{A}_V^1) \\ &= \text{Mor}(\varinjlim \mathfrak{Z}, \mathbb{A}_V^1) \\ &= \varprojlim \text{Mor}(\mathfrak{Z}, \mathbb{A}_V^1) = \varprojlim H_0(\mathfrak{Z}). \end{aligned}$$

\square

3.10. Lemma. *Let $R \subseteq S$ be an injective homomorphism of rings. Then the tensor square*

$$(10) \quad \begin{array}{ccc} R & \xrightarrow{\quad} & S \\ \downarrow & & \downarrow \\ S & \xrightarrow{\quad} & S \otimes_R S \end{array}$$

is Cartesian, that is to say, if $q \otimes 1 = 1 \otimes q$ in $S \otimes_R S$ for some $q \in S$, then in fact $q \in R$.

Proof. Let for simplicity assume that R and S are algebras over some field κ (since we only need the result for R and S Artinian, this already covers any equicharacteristic situation). Let $T := S \otimes_\kappa S$ be the tensor product over κ , and let \mathfrak{n} be the ideal in T generated by all expressions of the form $r \otimes 1 - 1 \otimes r$ for $r \in R$. Hence $S \otimes_R S \cong T/\mathfrak{n}$. If $q \in S$ satisfies $q \otimes 1 = 1 \otimes q$ in $S \otimes_R S$, then viewed as an element in T , the tensor $q \otimes 1 - 1 \otimes q$ lies in \mathfrak{n} . The canonical surjection $S \rightarrow S/R$ induces a homomorphism of tensor products $T \rightarrow (S/R) \otimes_\kappa (S/R)$. Under this homomorphism, \mathfrak{n} is sent to the zero ideal, whence so is in particular $q \otimes 1 - 1 \otimes q$. If the image of q in S/R were non-zero, then we can find a basis of S/R containing q . Hence $q \otimes 1$ and $1 \otimes q$ are two independent basis vectors of $(S/R) \otimes_\kappa (S/R)$, contradicting that they are equal in the latter ring. Hence $q \in R$, as we wanted to show. \square

By Lemma 3.1 and properties of tensor products, we have for any two sieves \mathfrak{X} and \mathfrak{Y} , a canonical homomorphism

$$(11) \quad H_0(\mathfrak{X}) \otimes_{\mathcal{O}_V} H_0(\mathfrak{Y}) \rightarrow H_0(\mathfrak{X} \times \mathfrak{Y}),$$

and a similar formula for algebraic sections. If \mathfrak{X} and \mathfrak{Y} are both representable, then this is an isomorphism, but not so in general.

The topos of sieves. The Zariski topology on a V -scheme X induces a *topos* on each of its sieves \mathfrak{X} . More precisely, the *admissible opens* on \mathfrak{X} are the sieves of the form $\mathfrak{X} \cap U^\circ$, where $U \subseteq X$ runs over all opens of X ; and the admissible coverings are all collections of admissible opens $\mathfrak{U}_i \subseteq \mathfrak{X}$ such that their union (as sieves) is equal to \mathfrak{X} , that is to say, such that the corresponding opens $U_i \subseteq X$ cover some ambient space of \mathfrak{X} . For simplicity, we will simply write $\mathfrak{X} \cap U$ for $\mathfrak{X} \cap U^\circ$. In particular, since X is quasi-compact, any admissible covering contains a finite admissible subcovering. The collection of admissible opens does not depend on the ambient space X , for if $X' \subseteq X$ is a locally closed subscheme on which \mathfrak{X} is also a sieve, then since its topology is induced by that of X , it induces the same admissible opens on \mathfrak{X} , and the same admissible coverings. Without going into details, we claim that the collection of admissible opens and admissible coverings yields a Grothendieck topology on \mathfrak{X} , turning $\mathbb{S}\text{ieve}_V$ into a Grothendieck site. Nonetheless, since for each fat point \mathfrak{z} , this induces a topological space on $\mathfrak{X}(\mathfrak{z})$, we will just pretend that we are working in a genuine topological space, and borrow the usual topological jargon.

3.11. Remark. Suppose V is the spectrum of a field κ . Unless \mathfrak{z} is the geometric point V itself, the topological space $\mathfrak{X}(\mathfrak{z})$ is not separated: two \mathfrak{z} -rational points $a, b \in \mathfrak{X}(\mathfrak{z})$ are inseparable if and only if they have the same center, that is to say, if and only if $\rho(\kappa)(a) = \rho(\kappa)(b)$, where $\rho(\kappa): \mathfrak{X}(\mathfrak{z}) \rightarrow \mathfrak{X}(\kappa)$ is the canonical map. Given an open $U \subseteq X$ with corresponding open sieve $\mathfrak{U} := \mathfrak{X} \cap U$, then, as we shall prove shortly in Theorem 9.3 below, we have $\mathfrak{U}(\mathfrak{z}) = \rho(\kappa)^{-1}(U(\kappa))$. Therefore, if $\rho(\kappa)(a) = \rho(\kappa)(b) \in U(\kappa)$, then $a, b \in \mathfrak{U}(\mathfrak{z})$.

3.12. Proposition. *If $s: \mathfrak{Y} \rightarrow \mathfrak{X}$ is an algebraic morphism of sieves, then it is continuous in the sense that the pull-back of any open in \mathfrak{X} is an open in \mathfrak{Y} .*

Proof. By assumption, s extends to a morphism $\varphi: Y \rightarrow X$, where \mathfrak{Y} is a sieve on Y and \mathfrak{X} on X . An open $\mathfrak{U} \subseteq \mathfrak{X}$ is of the form $\mathfrak{X} \cap U$, for some open subscheme $U \subseteq X$. Since $\varphi^{-1}(U)$ is open in Y and $\varphi^*\mathfrak{U} = \mathfrak{Y} \cap \varphi^{-1}(U)$, the claim follows. \square

To make $\mathbb{S}\text{ieve}_V$ into a topos, we need to define structure sheafs for a given sieve \mathfrak{X} . We define presheaves $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}}^{\text{geom}}$ on \mathfrak{X} , by associating to an open $\mathfrak{U} := \mathfrak{X} \cap U$ its \mathcal{O}_V -algebra of global sections

$$\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) := H_0(\mathfrak{X} \cap U) \quad \text{and} \quad \mathcal{O}_{\mathfrak{X}}^{\text{geom}}(\mathfrak{U}) := H_0^{\text{geom}}(\mathfrak{X} \cap U).$$

3.13. Corollary. *For each sieve \mathfrak{X} , the presheaves $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}}^{\text{geom}}$ are sheaves (in the topos sense).*

Proof. This is essentially the content of Theorem 3.5 (see also Remark 3.6), applied to a (finite) admissible covering of an open $\mathfrak{U} = \mathfrak{U}_1 \cup \dots \cup \mathfrak{U}_s$. \square

So we are justified in calling $\mathcal{O}_{\mathfrak{X}}$ the *structure sheaf* of the sieve \mathfrak{X} on a V -scheme X , and $\mathcal{O}_{\mathfrak{X}}^{\text{geom}}$ its *algebraic structure sheaf*.

Stalks. Let \mathfrak{X} be a sieve with ambient space X . A closed point $P \in X$ is called a *point* on \mathfrak{X} , if the closed immersion $i_P: P \subseteq X$, viewed as a P -rational point, belongs to $\mathfrak{X}(P)$, or equivalently, if $P^\circ \subseteq \mathfrak{X}$. We define the stalk at a point $P \in \mathfrak{X}$ as usual as the respective direct limits

$$\mathcal{O}_{\mathfrak{X},P} := \varinjlim \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \quad \text{and} \quad \mathcal{O}_{\mathfrak{X},P}^{\text{geom}} := \varinjlim \mathcal{O}_{\mathfrak{X}}^{\text{geom}}(\mathfrak{U})$$

where \mathfrak{U} runs over all admissible opens of \mathfrak{X} such that $P \in \mathfrak{U}$. Clearly, if $\mathfrak{X} = X^\circ$ is representable, then $\mathcal{O}_{X^\circ, P} = \mathcal{O}_{X^\circ, P}^{\text{geom}}$ is just the local ring $\mathcal{O}_{X, P}$ at the closed point $P \in X$ by Proposition 3.2. In fact, we have:

3.14. Proposition. *If \mathfrak{X} is a sieve which is Zariski dense in X , and if P is a point on \mathfrak{X} , then $\mathcal{O}_{\mathfrak{X}, P}^{\text{geom}} = \mathcal{O}_{X, P}$.*

Proof. One inclusion is immediate, so let $s \in \mathcal{O}_{\mathfrak{X}, P}^{\text{geom}}$. Hence there exists an open $U \subseteq X$ containing P such that $s: \mathfrak{X} \cap U \rightarrow \mathbb{A}_V^1$ is an algebraic section. Since $\mathfrak{X} \cap U$ is then Zariski dense in U , there exists an open ambient space $\tilde{U} \subseteq U$ of $\mathfrak{X} \cap U$ and a morphism $\tilde{U} \rightarrow \mathbb{A}_V^1$ extending s . This morphism corresponds to a global section of \tilde{U} and hence is an element in $\mathcal{O}_{\tilde{U}, P} = \mathcal{O}_{X, P}$, since \tilde{U} is open in X containing P . \square

More generally, if \mathfrak{X} is a sieve on X and P a point on \mathfrak{X} , then $\mathcal{O}_{\mathfrak{X}, P}^{\text{geom}} = \mathcal{O}_{\mathfrak{X}, P}$.

3.15. Lemma. *A global section $s: \mathfrak{X} \rightarrow \mathbb{A}_V^1$ of a sieve \mathfrak{X} is a unit if and only if the image of $s(P)$ does not contain zero, for any point $P \in \mathfrak{X}$. If V is the spectrum of an algebraically closed field κ , then this is equivalent with the image of $s(\kappa)$ not containing zero.*

Proof. One direction is clear, so assume that the image of $s(P)$ does not contain zero, for any closed point P . Let \mathfrak{z} be a fat point, and let P be its center. It follows from the commutative diagram

$$(12) \quad \begin{array}{ccc} \mathfrak{X}(\mathfrak{z}) & \xrightarrow{s(\mathfrak{z})} & R \\ \downarrow & & \downarrow \pi \\ \mathfrak{X}(P) & \xrightarrow{s(P)} & k(P) \end{array}$$

where $k(P)$ is the residue field of P , that the image of $s(\mathfrak{z})$ has empty intersection with the maximal ideal of the coordinate ring R of \mathfrak{z} , since π is just the residue map. Hence, for each $a \in \mathfrak{X}(\mathfrak{z})$, its image $s(\mathfrak{z})(a)$ is a unit in R , and hence we can define $t(\mathfrak{z})(a)$ to be its inverse. So remains to check that t is a morphism of sieves $\mathfrak{X} \rightarrow \mathbb{A}_V^1$, that is to say, a global section, and this is easy. \square

3.16. Proposition. *For each point P on a sieve \mathfrak{X} , the stalk $\mathcal{O}_{\mathfrak{X}, P}$ is a local ring.*

Proof. Let X be an ambient space of \mathfrak{X} . We have to show that given two non-units $s, t \in \mathcal{O}_{\mathfrak{X}, P}$, their sum is a non-unit as well. Shrinking X if necessary, we may assume that $s, t \in H_0(\mathfrak{X})$. I claim that $s(P)(i_P)$ and $t(i_P)(P)$ are both equal to zero, where $i_P: P \subseteq X$ is the closed immersion. Indeed, suppose not, say, $s(P)(i_P) \neq 0$, so that there exists an open $U \subseteq X$ containing P such that $s(Q)$ does not vanish on $U(Q)$ for any closed point $Q \in U$. By Lemma 3.15, this implies that s is a unit in $H_0(\mathfrak{X} \cap U)$ whence in $\mathcal{O}_{\mathfrak{X}, P}$, contradiction. Hence $s(P) + t(P)$ also vanishes at i_P and hence cannot be a unit in $\mathcal{O}_{\mathfrak{X}, P}$. Note that we in fact proved that the unique maximal ideal consists of all sections $s \in \mathcal{O}_{\mathfrak{X}, P}$ such that $s(P)(i_P) = 0$. \square

4. MOTIVIC SITES

As before, V is a fixed Noetherian, Jacobson scheme. A *motivic*³ site \mathbb{M} over V is a subcategory of \mathbf{Sieve}_V which is closed under products, and such that for any V -scheme X , the restriction $\mathbb{M}|_X$ (that is to say, the set of all \mathbb{M} -sieves on X) forms a lattice. In other words, if $\mathfrak{X}, \mathfrak{Y} \in \mathbb{M}$ are both sieves on a common scheme X , then $\mathfrak{X} \cap \mathfrak{Y}$ and $\mathfrak{X} \cup \mathfrak{Y}$ belong again to \mathbb{M} (including the minimum given by the empty set and the maximum given by X). We call \mathbb{M} an *explicit* motivic site, if all morphisms are algebraic. If \mathbb{M} is an arbitrary motivic site, then we let $\overline{\mathbb{M}}$ be the corresponding explicit motivic site, obtained by only taking algebraic morphisms.

Given a motivic site \mathbb{M} , as the sieves form locally a lattice (on each V -scheme), we can define its associate Grothendieck ring $\mathbf{Gr}(\mathbb{M})$ as the free Abelian group on symbols $\langle \mathfrak{X} \rangle$, where \mathfrak{X} runs over all \mathbb{M} -sieves, modulo the *scissor relations*

$$\langle \mathfrak{X} \rangle + \langle \mathfrak{Y} \rangle - \langle \mathfrak{X} \cup \mathfrak{Y} \rangle - \langle \mathfrak{X} \cap \mathfrak{Y} \rangle$$

for any two \mathbb{M} -sieves \mathfrak{X} and \mathfrak{Y} on a common V -scheme, and the *isomorphism relations*

$$\langle \mathfrak{X} \rangle - \langle \mathfrak{Y} \rangle$$

for any two \mathbb{M} -sieves \mathfrak{X} and \mathfrak{Y} that are \mathbb{M} -isomorphic. We denote the image of an \mathbb{M} -sieve \mathfrak{X} in $\mathbf{Gr}(\mathbb{M})$ by $[\mathfrak{X}]$. In particular, since each representable functor is in \mathbb{M} , we may associate to any V -scheme X its class $[X] := [X^\circ]$ in $\mathbf{Gr}(\mathbb{M})$. We define a multiplication on $\mathbf{Gr}(\mathbb{M})$ by the fiber product (one easily checks that this is well-defined): $[\mathfrak{X}] \cdot [\mathfrak{Y}] := [\mathfrak{X} \times \mathfrak{Y}]$. Since a motivic site has the same objects as its explicit counterpart, we get a canonical surjective homomorphism $\mathbf{Gr}(\overline{\mathbb{M}}) \rightarrow \mathbf{Gr}(\mathbb{M})$, which, however, need not be injective, since there are more isomorphism relations in the latter Grothendieck ring.

On occasion, we will encounter variants which are supported only on a subcategory of the category of all V -schemes (that is to say, we only require the restriction of the site to one of the schemes in the subcategory to be a lattice), and we can still associate a Grothendieck ring to it. We will refer to this as a *partial motivic site*. Most motivic sites \mathbb{M} will also have additional properties, like for instance being *stable under push-forwards along closed immersions*, meaning that if $i: Y \subseteq X$ is a closed subscheme and \mathfrak{Y} a motif in \mathbb{M} , then so is $i_*\mathfrak{Y}$. If this is the case, then \mathbb{M} is also closed under disjoint unions: given motives \mathfrak{X} and \mathfrak{X}' on X and X' respectively, then their disjoint union $\mathfrak{X} \sqcup \mathfrak{X}'$ is the union of the push-forwards $i_*\mathfrak{X}$ and $i'_*\mathfrak{X}'$, where $i: X \rightarrow X \sqcup X'$ and $i': X' \rightarrow X \sqcup X'$ are the canonical closed immersions.

Lefschetz class. The class of the affine line \mathbb{A}_V^1 plays a pivotal role in what follows; we call it the *Lefschetz class* and denote it by \mathbb{L} . On occasion, we will need to invert this class, and therefore consider localizations of the form $\mathbf{Gr}(\mathbb{M})_{\mathbb{L}}$.

5. THE SCHEMIC GROTHENDIECK RING

To connect the theory of motivic sites to the classical construction, we must describe motivic sites whose Grothendieck ring admits a natural homomorphism into the classical Grothendieck ring $\mathbf{Gr}(\mathbf{Var}_V)$ (obviously, this utterly fails for the motivic site of all sieves). We will first introduce the various motives of interest in the next few sections, before we settle this issue in Theorem 7.4 below. The smallest motivic site on V is obtained by taking

³We will refer to the objects in a motivic site as ‘motives’. This nomenclature is to express the fact that a motif represents something geometrical which is not a scheme but ought to be something like a scheme, thus ‘motivating’ our geometric treatment of it.

for sieves on a scheme X only the empty sieve and the whole sieve X° . The resulting Grothendieck ring has no non-trivial scissor relations and so we just get the free Abelian ring on isomorphism classes of V -schemes.

To define larger sites, we want to include at least closed subsieves of a scheme X . Any object in the lattice generated by the closed subsieves of X will be called a *schemic motif on X* . Since closed subsieves are already closed under intersection, a schemic motif on X is a sieve of the form

$$(13) \quad \mathfrak{X} = X_1^\circ \cup \cdots \cup X_s^\circ,$$

where the X_i are closed subschemes of X . Let us call a V -scheme X *schemically irreducible* if X° cannot be written as a finite union of proper closed subsieves. In particular, by an easy Noetherian argument, any schemic motif is the union of finitely many schemically irreducible closed subsieves. We call a representation (13) a *schemic decomposition*, if it is *irredundant*, meaning that there are no closed subscheme relations among any two X_i , and the X_i are schemically irreducible. Assume (13) is a schemic decomposition, and let $\mathfrak{X} = Y_1^\circ \cup \cdots \cup Y_t^\circ$ be a second schemic decomposition. Hence, for a fixed i , we have

$$X_i^\circ = X_i^\circ \cap \mathfrak{X} = (X_i \cap Y_1)^\circ \cup \cdots \cup (X_i \cap Y_t)^\circ.$$

Since X_i is schemically irreducible, there is some j such that $X_i = X_i \cap Y_j$, that is to say, $X_i \subseteq Y_j$. Reversing the roles of the two representations, the same argument yields some i' such that $Y_j \subseteq X_{i'}$. Since $X_i \subseteq Y_j \subseteq X_{i'}$, irredundancy implies that these are equalities. Hence, we proved:

5.1. Proposition. *A schemic decomposition is, up to order, unique.* \square

We call the X_i in the (unique) schemic decomposition (13) the *schemic irreducible components* of \mathfrak{X} . If X has dimension zero, then it is a finite disjoint sum of fat points, its schemic irreducible components. More generally, any schemic motif on X is a disjoint sum of closed subsieves given by fat points, and hence is itself a closed subsieve.

To give a purely scheme-theoretic characterization of being schemically irreducible, recall that a point $x \in X$ is called *associated*, if $\mathcal{O}_{X,x}$ has depth zero, that is to say, if every element in $\mathcal{O}_{X,x}$ is either a unit or a zero-divisor. Any minimal point is associated, and the remaining ones, which are also finite in number, are called *embedded*. The closure of an associated point is called a *primary component*. We say that X is *strongly connected*, if the intersection of all primary components is non-empty, that is to say, if there exists a (closed) point generalizing to each associated point (in the affine case $X = \text{Spec } A$, this means that the associated primes generate a proper ideal). For instance, if X is the union of two parallel lines and one intersecting line, then it is connected but not strongly. For an example with embedded points, take the affine line with two (embedded) double points given by the ideal $(x^2, xy(y-1))$. A (reduced) example where any two primary, but not all three, components meet, is given by the ‘triangle’ $xy(x-y-1)$.

5.2. Proposition. *A V -scheme X is schemically irreducible if and only if it is strongly connected.*

Proof. It is easier to work with the contrapositives of these statements, and we will show that their negations are then also equivalent with the existence of finitely many non-zero ideal sheafs $\mathcal{I}_1, \dots, \mathcal{I}_s \subseteq \mathcal{O}_X$ with the property that for each closed point x , there is some n such that $\mathcal{I}_n \mathcal{O}_{X,x} = 0$. To prove the equivalence of this with being schemically irreducible, assume $X^\circ = X_1^\circ \cup \cdots \cup X_s^\circ$ for some proper closed subschemes $X_n \subsetneq X$. Let \mathcal{I}_n be the ideal sheaf of X_n and let x be an arbitrary closed point. For each m , the closed

immersion $J_x^m X \subseteq X$ is a rational point on X along the m -th jet, whence must belong to one of the $X_n(J_x^m X)$, that is to say, $J_x^m X$ is a closed subscheme of X_n by Lemma 2.7. Since there are only finitely many possibilities, there is a single n such that each $J_x^m X$ is a closed subscheme of X_n . By Lemma 2.7, this means that $\mathcal{I}_n \mathcal{O}_{X,x}$ is contained in any power of the maximal ideal \mathfrak{m}_x , and hence by Krull's intersection theorem must be zero. Conversely, suppose there are non-zero ideal sheaves $\mathcal{I}_1, \dots, \mathcal{I}_s \subseteq \mathcal{O}_X$ such that at each closed point, at least one vanishes. Let X_n be the closed subscheme defined by \mathcal{I}_n , let \mathfrak{z} be a fat point, and let $a: \mathfrak{z} \rightarrow X$ be a \mathfrak{z} -rational point. Let x be the center of \mathfrak{z} , a closed point of X , and let $a_x: \mathcal{O}_{X,x} \rightarrow R$ be the induced local homomorphism on the stalks, where R is the coordinate ring of \mathfrak{z} . By assumption, there is some n such that $\mathcal{I}_n \mathcal{O}_{X,x} = 0$, whence so is its image under a_x . By Lemma 2.2, this implies that $a \in X_n(\mathfrak{z})$. Since this holds for any rational point, X° is the union of all X_n° .

We now prove the equivalence of the above condition with not being strongly connected. By (global) primary decomposition (see for instance [6, IV §3.2]), there exist (primary) closed subschemes $Y_n \subseteq X$ and an embedding $\mathcal{O}_X \hookrightarrow \mathcal{O}_{Y_1} \oplus \dots \oplus \mathcal{O}_{Y_s}$, such that the underlying sets of the Y_n are the primary components of X . Let \mathcal{J}_n be the ideal sheaf of Y_n , so that the above embeddability amounts to $\mathcal{J}_1 \cap \dots \cap \mathcal{J}_s = 0$. If X is not strongly connected, then the intersection of all Y_n is empty, which means that

$$(14) \quad \mathcal{J}_1 + \dots + \mathcal{J}_s = \mathcal{O}_X.$$

Let \mathcal{I}_j be the intersection of all \mathcal{J}_m with $m \neq j$, and let X_j be the closed subscheme given by \mathcal{I}_j . Let x be any closed point. By (14), we may assume after renumbering that the maximal ideal of x does not contain \mathcal{J}_1 , that is to say, $\mathcal{J}_1 \mathcal{O}_{X,x} = \mathcal{O}_{X,x}$. Therefore, $\mathcal{I}_1 \mathcal{O}_{X,x} = (\mathcal{I}_1 \cap \mathcal{J}_1) \mathcal{O}_{X,x}$, whence is zero, since $\mathcal{I}_1 \cap \mathcal{J}_1 = 0$. Conversely, assume there are non-zero ideal sheaves $\mathcal{I}_1, \dots, \mathcal{I}_s \subseteq \mathcal{O}_X$ such that $\mathcal{I}_n \mathcal{O}_{X,x} = 0$, for each closed point x and for some n depending on x . This is equivalent with the sum of all $\text{Ann}(\mathcal{I}_n)$ being the unit ideal. Since any annihilator ideal is contained in some associated prime, the sum of all associated primes must also be the unit ideal, and hence the intersection of all primary components is empty. \square

From the proof we learn that $\text{Spec } A$ is schemic reducible if and only if there exist finitely many proper ideals whose sum is the unit ideal and whose intersection is the zero ideal. We can even describe an algorithm which calculates its schemic irreducible components. Let $0 = \mathfrak{g}_1 \cap \dots \cap \mathfrak{g}_n$ be a primary decomposition of the zero ideal in A , and assume $\mathfrak{g}_1 + \dots + \mathfrak{g}_s = 1$, for some $s \leq n$. Then the schemic irreducible components of X are among the schemic irreducible components of the closed subschemes $X_i = \text{Spec}(A/\text{Ann}(\mathfrak{g}_i))$ for $i = 1, \dots, s$. That the X_i can themselves be schemic reducible, whence require further decomposition, is illustrated by the 'square' with equation $x(x-1)y(y-1) = 0$. A sufficient condition for the X_i to be already schemically irreducible is that $s = n$ and no fewer \mathfrak{g}_i generate the unit ideal. By Noetherian induction, this has to happen eventually. In the same vein, we have:

5.3. Proposition. *Suppose V is Jacobson. Let X be a V -scheme, and let $X_1, \dots, X_s \subseteq X$ be closed subschemes with respective ideals of definition $\mathcal{I}_1, \dots, \mathcal{I}_s \subseteq \mathcal{O}_X$. The Zariski closure of the schemic motif $\mathcal{X} := X_1^\circ \cup \dots \cup X_s^\circ$ is the closed subscheme with ideal of definition $\mathcal{I}_1 \cap \dots \cap \mathcal{I}_s$. In particular, \mathcal{X} is equal to its own Zariski closure if and only if $\mathcal{I}_1 + \dots + \mathcal{I}_s = \mathcal{O}_X$.*

Proof. Let Y be the closed subscheme with ideal of definition $\mathcal{I} := \mathcal{I}_1 \cap \dots \cap \mathcal{I}_s$. Since each $X_j \subseteq Y$, the Zariski closure of \mathcal{X} is contained in Y . To prove the converse, suppose

$Z \subseteq X$ is a closed subscheme such that $\mathfrak{X} \subseteq Z^\circ$. We have to show that $Y \subseteq Z$, so suppose not. This means that $\mathcal{J}\mathcal{O}_Y \neq 0$, where \mathcal{J} is the ideal of definition of Z . By the Jacobson condition, there is a closed point $x \in Y$ such that $\mathcal{J}\mathcal{O}_{Y,x} \neq 0$, and then by Krull's Intersection Theorem, some n such that $\mathcal{J}(\mathcal{O}_{Y,x}/\mathfrak{m}_x^n) \neq 0$, where \mathfrak{m}_x is the maximal ideal corresponding to x . We may write $\mathfrak{m}_x^n = \mathfrak{n}_1 \cap \dots \cap \mathfrak{n}_t$ as a finite intersection of irreducible ideal sheafs,⁴ and then for at least one, say for $j = 1$, we must have $\mathcal{J}(\mathcal{O}_{Y,x}/\mathfrak{n}_1) \neq 0$. Let \mathfrak{z} be the fat point with coordinate ring $R := \mathcal{O}_{Y,x}/\mathfrak{n}_1$. By Lemma 2.8, this means that the \mathfrak{z} -rational point i given by the inclusion $\mathfrak{z} \subseteq Y$ does not factor through Z . On the other hand, by the same Lemma, we have $\mathcal{I}R = 0$. Since the zero ideal is irreducible, at least one of the $\mathcal{I}_j R$ must vanish, showing that i lies in $\mathfrak{X}(\mathfrak{z})$, whence by assumption in $Z(\mathfrak{z})$, contradiction. The last assertion is now immediate from the previous discussion. \square

5.4. Corollary. *The global section ring of a schemic motif is equal to that of its Zariski closure.*

Proof. By an induction argument, we may reduce to the case that $\mathfrak{X} = Y^\circ \cup Z^\circ$ where $Y, Z \subseteq X$ are closed subschemes (alternatively, use (7)). In view of the local nature of the problem, we may furthermore reduce to the case that $X = \text{Spec } A$ is affine, so that Y and Z are defined by some ideals $I, J \subseteq A$. In particular, the Cartesian square (6) becomes

$$(15) \quad \begin{array}{ccc} H_0(\mathfrak{X}) & \xrightarrow{\quad} & A/I \\ \downarrow & & \downarrow \\ A/J & \xrightarrow{\quad} & A/(I+J). \end{array}$$

However, it is easy to check that putting $A/(I \cap J)$ in the left top corner of this square also yields a Cartesian square, and hence, by uniqueness, we must have $H_0(\mathfrak{X}) = A/(I \cap J)$. By Proposition 5.3, the Zariski closure of \mathfrak{X} is $\text{Spec}(A/(I \cap J))$, proving the assertion. \square

5.5. Definition. We define the *schemic motivic site* over V , denoted $\mathcal{S}\mathcal{C}\mathcal{H}_V$, as the full subcategory of $\mathcal{S}\mathcal{I}\mathcal{E}\mathcal{V}_V$ consisting of all schemic motives. By Proposition 2.7, the category of V -schemes fully embeds in $\mathcal{S}\mathcal{C}\mathcal{H}_V$, and the image of this embedding is precisely the full subcategory of representable schemic motives. In fact, by Theorem 3.7, all morphisms in $\mathcal{S}\mathcal{C}\mathcal{H}_V$ are algebraic, so that $\mathcal{S}\mathcal{C}\mathcal{H}_V$ is an explicit motivic site.

5.6. Lemma. *Any element in $\mathbf{Gr}(\mathcal{S}\mathcal{C}\mathcal{H}_V)$ is of the form $[X] - [Y]$, for some V -schemes X and Y .*

Proof. For any two V -schemes X and X' , we have $[X] + [X'] = [X \sqcup X']$, where $X \sqcup X'$ denotes their disjoint union. So remains to verify that any element in $\mathbf{Gr}(\mathcal{S}\mathcal{C}\mathcal{H}_V)$ is a linear combination of classes of schemes. This reduces the problem to the class of a single schemic motif \mathfrak{X} on a V -scheme X . Hence there exist closed subschemes $X_1, \dots, X_n \subseteq X$ such that $\mathfrak{X} = X_1^\circ \cup \dots \cup X_n^\circ$. For each non-empty $I \subseteq \{1, \dots, n\}$, let X_I be the closed subscheme obtained by intersecting all X_i with $i \in I$, and let $|I|$ denote the cardinality of I . A well-known argument deduces from the scissor relations the equality

$$(16) \quad [\mathfrak{X}] = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|} [X_I]$$

⁴An ideal is called *irreducible* if it cannot be written as a finite intersection of strictly larger ideals.

in $\mathbf{Gr}(\mathcal{S}\mathbf{ch}_V)$, proving the claim. \square

5.7. Theorem. *Suppose V is Jacobson. The spheric Grothendieck ring $\mathbf{Gr}(\mathcal{S}\mathbf{ch}_V)$ is freely generated, as a group, by the classes of strongly connected V -schemes.*

Proof. Let Γ be the free Abelian group generated by isomorphism classes $\langle X \rangle$ of strongly connected V -schemes X , and let Γ' be the free Abelian group generated by isomorphism classes $\langle \mathfrak{X} \rangle$ of spheric motives \mathfrak{X} . I claim that the composition $\Gamma \subseteq \Gamma' \twoheadrightarrow \mathbf{Gr}(\mathcal{S}\mathbf{ch}_V)$ admits an additive inverse $\delta: \mathbf{Gr}(\mathcal{S}\mathbf{ch}_V) \rightarrow \Gamma$.

To construct δ , we will first define an additive morphism $\delta': \Gamma' \rightarrow \Gamma$ which is the identity on Γ , and then argue that it vanishes on each scissor relation, inducing therefore a morphism $\delta: \mathbf{Gr}(\mathcal{S}\mathbf{ch}_V) \rightarrow \Gamma$. It suffices, by linearity, to define δ' on an isomorphism class of a spheric motif

$$(17) \quad \mathfrak{X} = X_1^\circ \cup \cdots \cup X_n^\circ,$$

given by a spheric decomposition with $X_i \subseteq X$ strongly connected closed subschemes. By Noetherian induction, we may furthermore assume that δ' has been defined on the isomorphism class of any spheric motif on a proper closed subscheme of X . In particular, $\delta'\langle X_I \rangle$ has already been defined, where we borrow the notation from (16). We therefore set

$$(18) \quad \delta'\langle \mathfrak{X} \rangle := \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|} \delta'\langle X_I \rangle$$

By Proposition 5.1, the spheric decomposition is unique, and hence δ' is well-defined. Moreover, if $n = 1$, so that $\mathfrak{X} = X_1^\circ$ is a schemically irreducible closed subscheme, whence strongly connected by Proposition 5.2, then its image under δ' is just $\langle X_1 \rangle$, showing that δ' is the identity on Γ .

Next, let us show that even if (17) is not irredundant, (18) still holds in Γ . We can go from such an arbitrary representation to the spheric decomposition in finitely many steps, by adding or omitting at each step one strongly connected closed subscheme contained in \mathfrak{X} . So assume (18) holds for a representation (17), and we now have to show that it also holds for the representation adding a $X_0^\circ \subseteq \mathfrak{X}$, with X_0 strongly connected. Since X_0 is schemically irreducible by Proposition 5.2, it must be a closed subscheme of one of the others, say, of X_1 . Let J range over all non-empty subsets of $\{0, \dots, n\}$. We arrange the subsets containing 0 in pairs $\{J_+, J_-\}$, so that J_+ and J_- differ only as to whether they contain 1 or not. Since $X_0 \subseteq X_1$, we get $X_{J_+} = X_{J_-}$, and as J_- has one element less than J_+ , the two terms indexed by this pair in the sum (18) cancel each other out. So, in that sum, only subsets J not containing 0 contribute, which is just the value for the representation without X_0° . We also have to consider the converse case, where instead we omit one, but the argument is the same.

We can now show that (18) is still valid even if the X_i in (17) are not strongly connected. Again we may reduce the problem to adding or omitting a single closed subscheme X_0° . Let $X_0^\circ = Y_1^\circ \cup \cdots \cup Y_m^\circ$ be a spheric decomposition for X_0 . We have to show that the value of the sum in (18) for the representation $\mathfrak{X} = X_0^\circ \cup \cdots \cup X_n^\circ$ is the same as that for the representation

$$(19) \quad \mathfrak{X} = X_1^\circ \cup \cdots \cup X_n^\circ \cup Y_1^\circ \cup \cdots \cup Y_m^\circ.$$

The first sum is given by

$$(20) \quad \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|} \delta'\langle X_I \rangle + \sum_{I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} \delta'\langle X_0 \cap X_I \rangle$$

By Noetherian induction and the fact that $(X_0 \cap X_I)^\circ = (X_I \cap Y_1)^\circ \cup \cdots \cup (X_I \cap Y_m)^\circ$, we have an identity

$$\delta' \langle X_0 \cap X_I \rangle = \sum_{\emptyset \neq J \subseteq \{1, \dots, m\}} (-1)^{|J|} \delta' \langle X_I \cap Y_J \rangle$$

for each subset I . Substituting this in (20) yields the sum corresponding to representation (19) (note that $I \cup J$ ranges over all non-empty subsets of $\{1, \dots, n\} \sqcup \{1, \dots, m\}$, as required).

To obtain the induced map δ , we must show next that δ' vanishes on any scissor relation. To this end, let $\mathfrak{Y} = Y_1^\circ \cup \cdots \cup Y_t^\circ$ be a second schemic motif on X , again assumed to be given by its schemic decomposition. Put $Z_{ij} := X_i \cap Y_j$, so that $\mathfrak{X} \cup \mathfrak{Y}$ is the union of the X_i° and Y_j° , whereas $\mathfrak{X} \cap \mathfrak{Y}$ is the union of the closed subsieves Z_{ij}° . By our previous argument, we may use these respective representations to calculate δ' of the scissor relation $\langle \mathfrak{X} \cup \mathfrak{Y} \rangle + \langle \mathfrak{X} \cap \mathfrak{Y} \rangle - \langle \mathfrak{X} \rangle - \langle \mathfrak{Y} \rangle$. Comparing the various sums given by the respective right hand sides of (18), this reduces to the following combinatorial assertion. Given finite subsets I, J , let us call a subset $N \subseteq I \times J$ *dominant*, if its projections onto the first and second coordinates are both surjective; then

$$\sum_{N \text{ dominant}} (-1)^{|N|} = 1.$$

We leave the details to the reader. In conclusion, we have constructed an (additive) map $\delta: \mathbf{Gr}(\mathcal{S}\mathbf{ch}_V) \rightarrow \Gamma$ which is the identity on Γ . On the other hand, it follows from (16) that $[\delta[\mathfrak{X}]] = [\mathfrak{X}]$, showing that δ is an isomorphism. \square

Immediately from this we get:

5.8. Corollary. *Suppose V is Jacobson. If X and Y are strongly connected V -schemes, then $[X] = [Y]$ in $\mathbf{Gr}(\mathcal{S}\mathbf{ch}_V)$ if and only if $X \cong Y$.* \square

6. THE SUB-SCHEMIC GROTHENDIECK RING

To allow for additional relations, we want to include also open subsieves, or more generally, locally closed subsieves. For applications, it is more appropriate to put this in a larger context. Our point of departure is:

6.1. Lemma. *For any V -scheme, the set of its sub-schemic sieves forms a lattice. Moreover, the product of two sub-schemic sieves is again sub-schemic.*

Proof. Recall that a sub-schemic sieve is just an image sieve. If $\varphi: Y \rightarrow X$ and $\psi: Z \rightarrow X$ are morphisms of V -schemes, then $\mathrm{Im}(\varphi) \cap \mathrm{Im}(\psi) = \mathrm{Im}(\varphi \times_X \psi)$, where $\varphi \times_X \psi: Y \times_X Z \rightarrow X$ is the total morphism in the commutative square

$$(21) \quad \begin{array}{ccc} Y \times_X Z & \xrightarrow{\quad} & Y \\ \downarrow & & \downarrow \varphi \\ Z & \xrightarrow{\quad \psi \quad} & X \end{array}$$

given by base change. Likewise $\mathrm{Im}(\varphi) \cup \mathrm{Im}(\psi) = \mathrm{Im}(\varphi \sqcup \psi)$, where $\varphi \sqcup \psi: Y \sqcup Z \rightarrow X$ is the disjoint union of the two morphisms.

As for products, if $\varphi: Y \rightarrow X$ and $\varphi': Y' \rightarrow X'$ are morphisms of V -schemes, then the image of $\varphi \times_V \varphi': Y \times_V Y' \rightarrow X' \times_V X$ is equal to the product $\mathrm{Im}(\varphi) \times \mathrm{Im}(\varphi')$, showing that the latter is again sub-schematic. \square

6.2. Definition. We define the *sub-schematic motivic site* $\mathrm{subS\acute{c}h}_V$ as the full subcategory of Siev_V with objects the sub-schematic sieves. By Theorem 3.7, any morphism in this category is algebraic, so that $\mathrm{subS\acute{c}h}_V$ is again an explicit motivic site.

Instead, we could have opted for a smaller site to take care of open coverings: define the *motivic constructible site* Con_V by taking on each scheme the lattice generated by locally closed subsieves (note that this is again an explicit motivic site). We have a natural homomorphism of Grothendieck rings $\mathrm{Gr}(\mathrm{Con}_V) \rightarrow \mathrm{Gr}(\mathrm{subS\acute{c}h}_V)$, but I do not know whether it is injective and/or surjective.

6.3. Example. The constructible site is strictly smaller than the sub-schematic one, as illustrated by the following example: let $\varphi: \mathbb{A}_1 \rightarrow \mathbb{A}_1$ be the morphism corresponding to the homomorphism $R_2 := \kappa[\xi]/\xi^2\kappa[\xi] \rightarrow R_4 := \kappa[\xi]/\xi^4\kappa[\xi]$ given by $\xi \mapsto \xi^2$. Given a fat point $\mathfrak{z} = \mathrm{Spec} R$, the \mathfrak{z} -rational points of \mathbb{A}_1 are in one-one correspondence with the elements in R whose square is zero, whereas $\mathrm{Im}(\varphi)(\mathfrak{z})$ is the subset of all those that are themselves a square, in general a proper subset. As \mathbb{A}_1 is a fat point, it has no non-trivial locally closed subsieves, showing that $\mathrm{Im}(\varphi)$ is sub-schematic but not constructible. Moreover, the Zariski closure of $\mathrm{Im}(\varphi)$ is \mathbb{A}_1 .

6.4. Lemma. *If Y is an open in a V -scheme X , then Y° is a complete sieve on X .*

Proof. Let $\mathfrak{v} \rightarrow \mathfrak{w}$ be a morphism of fat points. We have to show that any \mathfrak{w} -rational point $a: \mathfrak{w} \rightarrow X$ whose image under $X(\mathfrak{w}) \rightarrow X(\mathfrak{v})$ belongs to $Y(\mathfrak{v})$, itself already belongs to $Y(\mathfrak{w})$. The condition that needs to be checked is that if the composition $\mathfrak{v} \rightarrow \mathfrak{w} \xrightarrow{a} X$ factors through Y , then so does a . Let $x \in X$ be the center of a . Since x is then also the center of the composition $\mathfrak{v} \rightarrow X$, it is a closed point of Y . Therefore, $\mathcal{O}_{X,x} = \mathcal{O}_{Y,x}$. Since a induces a homomorphism $\mathcal{O}_{X,x} \rightarrow T$, where T is the coordinate ring of \mathfrak{w} , whence a homomorphism $\mathcal{O}_{Y,x} \rightarrow T$, we get the desired factorization $\mathfrak{w} \rightarrow Y$. \square

This will, among other things, allow us often to reduce the calculation of rational points to the affine case. Let $X = X_1 \cup \dots \cup X_n$ be an open cover. By Lemma 6.4, we get $X^\circ = X_1^\circ \cup \dots \cup X_n^\circ$. An easy argument on scissor relations, with notation as in (16), yields:⁵

6.5. Lemma. *If $X = X_1 \cup \dots \cup X_n$ is an open covering of V -schemes, then*

$$[X] = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|} [X_I]$$

in $\mathrm{Gr}(\mathrm{subS\acute{c}h}_V)$. In particular, the class of a schematic motif lies in the subring generated by classes of affine schemes. \square

6.6. Example. As an example, let us calculate the class of the projective line \mathbb{P}_V^1 . It admits an open covering $X_1 \cup X_2$ where X_1 and X_2 are obtained by removing respectively the origin and the point at infinity. Since $X_1 \cong X_2 \cong \mathbb{A}_V^1$, we have

$$[\mathbb{P}_V^1] = 2\mathbb{L} - \mathbb{L}_*$$

where \mathbb{L}_* denotes the class of the punctured line $L_* = X_1 \cap X_2$, the affine line with the origin removed. One would be tempted to think that \mathbb{L}_* is just $\mathbb{L} - 1$, but this is false, as

⁵By assumption, all V -schemes are separated, and hence the intersection of affines is again affine.

we shall see shortly. However, $X_1 \cap X_2$ is an affine scheme, by Example 2.10, isomorphic to the hyperbola $H \subseteq \mathbb{A}_V^2$ with equation $xy - 1 = 0$ under the projection $\mathbb{A}_V^2 \rightarrow \mathbb{A}_V^1$ onto the first coordinate. In other words, we have

$$(22) \quad [\mathbb{P}_V^1] = 2\mathbb{L} - [H].$$

7. THE FORMAL GROTHENDIECK RING

Let $Y \subseteq X$ be a closed subscheme. Recall that the n -th jet of X along Y , denoted $J_Y^n X$, is the closed subscheme with ideal sheaf \mathcal{I}_Y^n , where $\mathcal{I}_Y \subseteq \mathcal{O}_X$ is the ideal sheaf of Y . The *formal completion* \hat{X}_Y of X along Y is then the ringed space whose underlying set is equal to the underlying set of Y and whose sheaf of rings is the inverse limit of the sheaves $\mathcal{O}_{J_Y^n X}$. In particular, if $X = \text{Spec } A$ is affine and I the ideal of definition of Y , then the ring of global sections of \hat{X}_Y is equal to the I -adic completion \hat{A} of A (see, for instance, [7, II.§9]). We define the *completion sieve along Y* to be the sieve $\text{Mor}_V(\cdot, \hat{X}_Y)$ represented by the formal completion \hat{X}_Y of X at Y , that is to say, for each fat point \mathfrak{z} , it gives the subset of all \mathfrak{z} -rational points $\mathfrak{z} \rightarrow X$ that factor through \hat{X}_Y . We will simply denote it by \hat{X}_Y° and call any such presheaf again *pro-representable*.⁶

7.1. Proposition. *For a closed subscheme $Y \subseteq X$, the completion sieve \hat{X}_Y° of X along Y is equal to the union of all closed subsieves $(J_Y^n X)^\circ \subseteq X^\circ$, for $n \geq 1$. Moreover, we have an identity of sieves $\hat{X}_Y^\circ = X^\circ - (X - Y)^\circ$, showing that \hat{X}_Y° is a complete sieve.*

Proof. The inclusion $J_Y^n X \subseteq \hat{X}_Y^\circ$, for any n , is clear since the jets of \hat{X}_Y (with respect to its closed point) are the same as those of the germ (X, Y) . Let \mathfrak{z} be a fat point of length l . If $a: \mathfrak{z} \rightarrow \hat{X}_Y$ is a \mathfrak{z} -rational point, then this must already factor through $J_Y^l X$, as any l -th power of a non-invertible section on \mathfrak{z} is zero. Hence $(J_Y^l X)^\circ \subseteq \hat{X}_Y^\circ$ have the same \mathfrak{z} -rational points, proving the first assertion. Since $J_Y^l X$ has the same underlying variety as Y , it lies outside the open $X - Y$, and hence $a \notin (X - Y)(\mathfrak{z})$. To prove the converse inclusion in the second assertion, suppose now that $a: \mathfrak{z} \rightarrow X$ does not lie in $(X - Y)(\mathfrak{z})$. In particular, the center of a lies in Y . Let $\text{Spec } A$ be an affine open of X containing the center of a , and let (R, \mathfrak{m}) be the (Artinian local) coordinate of \mathfrak{z} . Hence a induces a \mathcal{O}_U -algebra homomorphism $A \rightarrow R$. If I is the ideal locally defining Y in $\text{Spec } A$, then $IR \subseteq \mathfrak{m}$. In particular, $I^l R \subseteq \mathfrak{m}^l = 0$, showing that a lies in $(J_Y^l X)(\mathfrak{z})$ whence in $\hat{X}_Y(\mathfrak{z})$ by our first inclusion. The completeness of \hat{X}_Y° now follows from Lemma 2.5. \square

The proof of the first assertion actually gives a stronger statement, which we formalize as follows. A sieve \mathfrak{X} on X is called a *formal motif on X* , if for each fat point \mathfrak{z} , there exists a sub-schemic subsieve $\mathfrak{Y}_{\mathfrak{z}} \subseteq \mathfrak{X}$ such that $\mathfrak{Y}_{\mathfrak{z}}(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z})$ (we call the $\mathfrak{Y}_{\mathfrak{z}}$ the *sub-schemic approximations* of \mathfrak{X} , in spite of the fact that they are not unique). A sub-schemic motif is a trivial example of a formal motif; the proof of Proposition 7.1 shows that completion sieves are formal too, whose approximations, in fact, can be taken to be schemic. More generally, we have

7.2. Lemma. *If a sieve \mathfrak{X} on a V -scheme X has formal approximations in the sense that for each fat point \mathfrak{z} , there exists a formal subsieve $\mathfrak{Y}_{\mathfrak{z}} \subseteq \mathfrak{X}$ with the same \mathfrak{z} -rational points, then \mathfrak{X} itself is formal.*

⁶Note that \hat{X}_Y is no longer a scheme, but only a locally ringed space with values in the category of \mathcal{O}_V -algebras, and so for a (formal) scheme Z , the set $\text{Mor}_V(Z, \hat{X}_Y)$ is to be understood as the set of morphisms $Z \rightarrow \hat{X}_Y$ of locally ringed spaces with values in the category of \mathcal{O}_V -algebras.

Proof. By assumption, there exists a sub-schematic approximation $\mathfrak{Z}_3 \subseteq \mathfrak{Y}_3$ with the same \mathfrak{z} -rational points, and it is now easy to check that the \mathfrak{Z}_3 form a sub-schematic approximation of \mathfrak{X} . \square

7.3. Lemma. *If a formal motif \mathfrak{X} has no \mathfrak{z} -rational points, for some fat point \mathfrak{z} , then \mathfrak{X} itself is empty.*

Proof. Suppose first that $\mathfrak{X} = \mathrm{Im}(\varphi)$ is sub-schematic, given by a morphism $\varphi: Y \rightarrow X$. In order for $\varphi(\mathfrak{z})$ to be empty, $Y(\mathfrak{z})$ has to be empty, whence Y has to be the empty scheme by Lemma 2.8, proving that $\mathrm{Im}(\varphi)$ is the empty motif. Suppose now that \mathfrak{X} is merely formal, and let \mathfrak{w} be an arbitrary fat point. Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a sub-schematic approximation with the same \mathfrak{w} -rational points. Since $\mathfrak{Y}(\mathfrak{z}) \subseteq \mathfrak{X}(\mathfrak{z}) = \emptyset$, we get $\mathfrak{Y} = \emptyset$, by the sub-schematic case, showing that $\mathfrak{X}(\mathfrak{w}) = \emptyset$. Since this holds for all fat points \mathfrak{w} , the assertion follows. \square

Using Lemma 6.1, one easily verifies that the formal motives on X form again a lattice, and the product of two formal motives is again a formal motif, leading to the *formal motivic site* Form_V , and its corresponding Grothendieck ring $\mathrm{Gr}(\mathrm{Form}_V)$.

7.4. Theorem. *Over an algebraically closed field κ , we have natural ring homomorphisms*

$$\mathrm{Gr}(\mathrm{Sch}_\kappa) \rightarrow \mathrm{Gr}(\mathrm{Con}_\kappa) \rightarrow \mathrm{Gr}(\mathrm{subSch}_\kappa) \rightarrow \mathrm{Gr}(\mathrm{Form}_\kappa) \rightarrow \mathrm{Gr}(\mathrm{Var}_\kappa).$$

Proof. Only the last of these homomorphisms requires an explanation. Given a formal motif \mathfrak{X} , we associate to it the class of $\mathfrak{X}(\kappa)$ in the classical Grothendieck ring $\mathrm{Gr}(\mathrm{Var}_\kappa)$. Note that by definition, $\mathfrak{X}(\kappa) = \mathrm{Im}(\varphi)(\kappa)$, for some morphism $\varphi: Y \rightarrow X$ of κ -schemes. In particular, by Chevalley's theorem, $\mathfrak{X}(\kappa)$ is a constructible subset of $X(\kappa)$ and hence its class in $\mathrm{Gr}(\mathrm{Var}_\kappa)$ is well-defined. Clearly, this map is compatible with intersections, unions, and products, so that in order for this map to factor through $\mathrm{Gr}(\mathrm{Form}_\kappa)$, we only have to show that it respects isomorphisms. So assume $s: \mathfrak{X} \rightarrow \mathfrak{Y}$ is an isomorphism of formal motives. Let $\mathfrak{Z} \subseteq \mathfrak{X}$ be a sub-schematic approximation of \mathfrak{X} with the same κ -rational points. Its push-forward $s_*\mathfrak{Z}$ is isomorphic with \mathfrak{Z} . By Theorem 3.7, the restriction $s|_{\mathfrak{Z}}$ extends to a morphism $\varphi: X \rightarrow Y$, where X and Y are some ambient spaces of \mathfrak{Z} and \mathfrak{Y} respectively. Since $\varphi(\kappa): X(\kappa) \rightarrow Y(\kappa)$ maps $\mathfrak{Z}(\kappa)$ bijectively onto $s_*\mathfrak{Z}(\kappa)$, these two constructible subsets are isomorphic in the Zariski topology. However, by definition of push-forward, $s_*\mathfrak{Z}(\kappa)$ is the image of $\mathfrak{Z}(\kappa) = \mathfrak{X}(\kappa)$ under $s(\kappa)$, that is to say, is equal to $\mathfrak{Y}(\kappa)$, as we needed to show. \square

7.5. Theorem. *Assume V is Jacobson, and let $s: \mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism of sieves. If \mathfrak{Y} and \mathfrak{X} are sub-schematic (respectively, formal) motives, then so is the graph of s . Moreover, the pull-back or the push-forward of a sub-schematic (respectively, formal) submotif is again of that form.*

Proof. Let X be an ambient spaces of \mathfrak{X} . Since the graph of the composition $\mathfrak{Y} \rightarrow \mathfrak{X} \subseteq X^\circ$ is equal to the intersection of the graph $\Gamma(s)$ of s with $\mathfrak{Y} \times \mathfrak{X}$, we may assume from the start that $\mathfrak{X} = X^\circ$. Assume first that \mathfrak{Y} is sub-schematic. By Theorem 3.7, the morphism s extends to a morphism $\varphi: Y \rightarrow X$ of V -schemes. Let $Z \subseteq Y \times_V X$ be the graph of this morphism, which therefore is a closed subscheme. Since $\Gamma(s)$ is equal to the intersection $Z^\circ \cap (\mathfrak{Y} \times X^\circ)$, it is again sub-schematic.

Suppose next that \mathfrak{Y} is merely a formal motif, and, for each fat point \mathfrak{z} , let $\mathfrak{Z}_3 \subseteq \mathfrak{Y}$ be a sub-schematic approximation. Since $\Gamma(s)$ contains the graph of the restriction of s to \mathfrak{Z}_3 , and since they have the same \mathfrak{z} -rational points, the assertion follows from what we just proved for sub-schematic motives.

Let $\mathfrak{Y}' \subseteq \mathfrak{Y}$ be a sub-schematic or formal submotif. Since the push-forward $s_*\mathfrak{Y}$ is the image of the restriction of s to \mathfrak{Y}' , we may reduce the problem to showing that $\mathrm{Im}(s)$ is respectively sub-schematic or formal. The formal case follows easily, as in the previous argument, from the sub-schematic one. So assume once more that \mathfrak{Y} is sub-schematic, say of the form, $\mathrm{Im}(\psi)$ with $\psi: Z \rightarrow Y$ a morphism of V -schemes. With φ as above, one easily verifies that $\mathrm{Im}(s) = \mathrm{Im}(\varphi \circ \psi)$. To prove the same for the pull-back, simply observe that the pull-back $s^*\mathfrak{X}'$ of a submotif $\mathfrak{X}' \subseteq \mathfrak{X}$ is equal to the image of $\Gamma(s) \cap (\mathfrak{Y} \times \mathfrak{X}')$ under the morphism induced by the projection $Y \times X \rightarrow Y$. The result then follows from our previous observations. \square

7.6. Corollary. *Given an irreducible closed subscheme $Y \subseteq X$ with ideal of definition \mathcal{I}_Y , let \bar{X} be the closed subscheme given by the intersection of all powers \mathcal{I}_Y^n (e.g., if X is integral, this is just X itself, by Krull's Intersection Theorem). We have isomorphisms of \mathcal{O}_V -algebras*

$$H_0^{\mathrm{geom}}(\hat{X}_Y^\circ) \cong \mathcal{O}_{\bar{X},y} \quad \text{and} \quad H_0(\hat{X}_Y^\circ) \cong \hat{\mathcal{O}}_{\bar{X},y}$$

where y is the generic point of Y .

Proof. Let $\mathfrak{X} := \hat{X}_Y^\circ$ be the formal motif determined by the formal completion along Y . We leave it to the reader to verify that the Zariski closure of \mathfrak{X} is equal to \bar{X} . Replacing X by \bar{X} , we may therefore assume that \mathfrak{X} is Zariski dense. An open subscheme $U \subseteq \bar{X}$ is an ambient space of \mathfrak{X} if and only if $Y \subseteq U$, since $\mathfrak{X}(\mathfrak{z}) \subseteq Y(\mathfrak{z})$, for each fat point \mathfrak{z} . By Corollary 3.3, the ring of algebraic sections $H_0^{\mathrm{geom}}(\mathfrak{X})$ is therefore the direct limit of all $H_0(U)$, where $U \subseteq \bar{X}$ runs over all opens containing Y . The latter condition is equivalent with $y \in U$, and hence this direct limit is just $\mathcal{O}_{\bar{X},y}$. On the other hand, since the jets $J_Y^n X$ are approximations of \mathfrak{X} , the inverse limit of the $H_0(J_Y^n X)$ is equal to $H_0(\mathfrak{X})$ by Lemma 3.9. Since $H_0(J_Y^n X) = \Gamma(\mathcal{O}_X/\mathcal{I}_Y^n, X)$, the result follows. \square

7.7. Example. Let $\hat{\mathbb{L}}$ be the class of the formal completion $\hat{\mathbb{L}}$ of the affine line along the origin. It follows from Proposition 7.1 that $\hat{\mathbb{L}} = \mathbb{L} - \mathbb{L}_*$ (see Example 6.6). In particular, (22) becomes

$$[\mathbb{P}_V^1] = \mathbb{L} + \hat{\mathbb{L}}.$$

We will shortly generalize this in Proposition 7.8 below, but let us first construct from this an example of a non-algebraic morphism. Let $f(x)$ be a power series in a single indeterminate which is not a polynomial. The homomorphism $\kappa[t] \rightarrow \kappa[[x]]$ given by $t \mapsto f$ induces a natural transformation of sieves $s_f: \hat{L}^\circ \rightarrow (\mathbb{A}_\kappa^1)^\circ$. Since f is not a polynomial, it cannot extend to a morphism of schemes. Its graph, in accordance with Theorem 7.5, is the formal motif with approximations the graphs of the algebraic morphisms given by the various truncations of f .

We can also use this to give a counterexample to Proposition 3.12 for global sections. In general, given a global section $s: \mathfrak{X} \rightarrow \mathbb{A}_V^1$ of a formal motif \mathfrak{X} , define a sieve \mathfrak{X}_s by letting $\mathfrak{X}_s(\mathfrak{z})$ consist of all \mathfrak{z} -rational points $a \in \mathfrak{X}(\mathfrak{z})$ such that $s(\mathfrak{z})(a)$ is a unit, for each fat point \mathfrak{z} . Since $\mathfrak{X}_s = s^*L_*^\circ$, where L_* is the affine line without the origin, it is a formal motif by Theorem 7.5. Applied to the global section s_f above, $\hat{L}_{s_f}^\circ$ is the intersection of the open subsieves given by the truncations of f , whence not an admissible open in \mathfrak{X} for the Zariski topos. In particular, s_f is not continuous. Put differently, the submotives of the form \mathfrak{X}_s form in general a basis for a Grothendieck topology which is stronger than the Zariski one.

7.8. Proposition. *For each n , the class of projective n -space in $\mathbf{Gr}(\mathbf{Form}_V)$ is given by the formula*

$$[\mathbb{P}_V^n] = \sum_{m=0}^n \mathbb{L}^m \cdot \hat{\mathbb{L}}^{n-m}.$$

Proof. Let $(x_0 : \cdots : x_n)$ be the homogeneous coordinates of \mathbb{P}_V^n , and let X_i be the basic open given as the complement of the x_i -hyperplane. Hence every X_i is isomorphic with \mathbb{A}_V^n and their union is equal to \mathbb{P}_V^n . Therefore, by Lemma 6.5, we have

$$(23) \quad [\mathbb{P}_V^n] = \sum_{\emptyset \neq I \subseteq \{0, \dots, n\}} (-1)^{|I|} [X_I]$$

in $\mathbf{Gr}(\mathbf{Form}_V)$. So we need to calculate the class of each X_I . One easily verifies that, for $m \geq 0$, any intersection of m different opens X_i is isomorphic to the open $\mathbb{A}_V^{n-m} \times (L_*)^m$, where L_* is the affine line \mathbb{A}_V^1 minus a point. Since $[L_*] = \mathbb{L}_* = \mathbb{L} - \hat{\mathbb{L}}$ by Proposition 7.1, the class of such an intersection is equal to the product $\mathbb{L}^{n-m}(\mathbb{L} - \hat{\mathbb{L}})^m$. Since there are $\binom{n+1}{m}$ terms with $|I| = m$ in (23), the class of \mathbb{P}_V^n is equal to $g(\mathbb{L}, \hat{\mathbb{L}})$, where

$$g(t, u) := \sum_{m=0}^n (-1)^m \binom{n+1}{m} t^{n-m} (t-u)^m.$$

By the binomial theorem, $t^{n+1} - (t-u)g(t, u) = (t - (t-u))^{n+1} = u^{n+1}$, and hence

$$g(t, u) = \frac{t^{n+1} - u^{n+1}}{t-u} = \sum_{m=0}^n t^m u^{n-m},$$

as we wanted to show. \square

We conclude this section with a characterization of the complete sieves among the formal motives over an algebraically closed field κ .

7.9. Theorem. *Let κ be an algebraically closed field. A sieve \mathfrak{X} on a κ -scheme X is a complete formal motif if and only if it is the cone $\mathfrak{C}(F)$ over a constructible subset $F \subseteq X(\kappa)$.*

Proof. Suppose \mathfrak{X} is complete and formal. By Lemma 2.6, it is of the form $\mathfrak{C}(F)$ for some $F \subseteq X(\kappa)$. In fact, $F = \mathfrak{X}(\kappa)$, and hence by definition of the form $\mathrm{Im}(\varphi)(\kappa)$ for some morphism $\varphi: Y \rightarrow X$. By Chevalley's theorem, F is constructible. To prove the converse, since cones are easily seen to commute with union and intersection, and since any constructible subset of $\mathfrak{X}(\kappa)$ is an intersection and union of closed and open subsets, it suffices to prove that $\mathfrak{C}(F)$ is formal, whenever F is a closed or an open subset. The open case follows immediately from Lemma 6.4, and in the closed case, we have $\mathfrak{C}(F) = \hat{X}_F^\circ$ by Proposition 7.1 (note that the completion of a scheme along a subscheme only depends on the underlying variety of the subscheme, so that \hat{X}_F is well-defined). \square

8. ADJUNCTION

Let V and W be two Noetherian Jacobson schemes. By a (*schemic*) *adjunction* over the pair (V, W) , we mean a pair of functors $\eta: \mathbf{Fat}_W \rightarrow \mathbf{Fat}_V$ and $\nabla: \mathbf{Sch}_V \rightarrow \mathbf{Sch}_W$, called respectively the *left* and *right adjoint*, such that we have, for each W -fat point \mathfrak{z} and each V -scheme X , an adjunction isomorphism

$$(24) \quad \Theta_{\mathfrak{z}, X}: X(\eta(\mathfrak{z})) = \mathrm{Mor}_V(\eta(\mathfrak{z}), X) \cong \mathrm{Mor}_W(\mathfrak{z}, \nabla X) = \nabla X(\mathfrak{z}),$$

which is functorial in both arguments. Whenever \mathfrak{z} and X are clear from the context, we may just denote this isomorphism by Θ , or even omit it altogether, thus identifying $X(\eta(\mathfrak{z}))$ with $\nabla X(\mathfrak{z})$. More generally, by an (arbitrary) *adjunction* we mean the same as above, except that the right adjoint now only takes values in the category of sieves, that is to say, is a functor $\mathcal{S}\mathcal{c}\mathcal{h}_V \rightarrow \mathcal{S}\mathcal{i}\mathcal{e}\mathcal{v}\mathcal{e}_W$, where we identify the category of V -schemes with its image as a full subcategory of sieves. Of course, the morphisms on the right hand side of (24) are now to be taken in $\mathcal{S}\mathcal{i}\mathcal{e}\mathcal{v}\mathcal{e}_W$, where the last equality is then given by Corollary 2.9 (note, however, that they are all algebraic by Proposition 2.7). If each ∇X is sub-schematic, or formal, then we call the adjunction respectively *sub-schemic* or *formal*.

We can formulate the adjunction property as a representability question: given a functor $\eta: \mathcal{F}\mathcal{a}\mathcal{t}_W \rightarrow \mathcal{F}\mathcal{a}\mathcal{t}_V$ and a V -sieve \mathfrak{X} , let $\nabla_\eta \mathfrak{X}$ be the presheaf over W associating to a fat W -point, the set $\mathfrak{X}(\eta(\mathfrak{z}))$. We have adjunction when all presheaves $\nabla_\eta X^\circ$ are sieves, where X varies over all V -schemes; the adjunction is (sub-)schemic or formal, if each $\nabla_\eta X^\circ$ is respectively a (sub-)schemic or formal motif. From this perspective, ∇_η is the right adjoint of η , and we simply call ∇_η the adjunction. We extend this to get a functor $\nabla_\eta: \mathcal{S}\mathcal{i}\mathcal{e}\mathcal{v}\mathcal{e}_V \rightarrow \mathcal{S}\mathcal{i}\mathcal{e}\mathcal{v}\mathcal{e}_W$ as follows. Given a sieve \mathfrak{X} on a V -scheme X , we define its *adjoint* $\nabla_\eta \mathfrak{X}$ as the presheaf over W given by

$$\nabla_\eta \mathfrak{X}(\mathfrak{z}) := \Theta_{\mathfrak{z}, X}(\mathfrak{X}(\eta(\mathfrak{z})))$$

for any W -point \mathfrak{z} . It follows immediately from (24) that $\nabla_\eta X^\circ = \nabla X$, and hence $\nabla_\eta \mathfrak{X}$ is a subsieve of ∇X . The adjunction isomorphism (24) then becomes

$$(25) \quad \text{Mor}_{\mathcal{S}\mathcal{i}\mathcal{e}\mathcal{v}\mathcal{e}_V}(\eta(\mathfrak{z}), \mathfrak{X}) \cong \text{Mor}_{\mathcal{S}\mathcal{i}\mathcal{e}\mathcal{v}\mathcal{e}_W}(\mathfrak{z}, \nabla_\eta \mathfrak{X}).$$

8.1. Lemma. *If $\varphi: Y \rightarrow X$ is a morphism of V -schemes, then, with $\nabla\varphi: \nabla Y \rightarrow \nabla X$ the induced morphism of W -sieves, we have an equality of sieves*

$$(26) \quad \nabla_\eta \text{Im}(\varphi) = \text{Im}(\nabla\varphi).$$

In particular, sub-schemic adjunctions preserve sub-schemic as well as formal motives, whereas formal adjunctions preserve formal motives.

Proof. We verify (26) on a W -fat point \mathfrak{z} . Functoriality of adjunction implies that we have a one-one correspondence of diagrams

$$(27) \quad \begin{array}{ccc} & & Y \\ & \nearrow b & \downarrow \varphi \\ \eta(\mathfrak{z}) & & X \\ & \searrow a & \\ & & \end{array} \quad \xrightleftharpoons[\Theta^{-1}]{\Theta} \quad \begin{array}{ccc} & & \nabla Y \\ & \nearrow \tilde{b} & \downarrow \nabla\varphi \\ \mathfrak{z} & & \nabla X \\ & \searrow \tilde{a} & \end{array}$$

where the right triangle is in $\mathcal{S}\mathcal{i}\mathcal{e}\mathcal{v}\mathcal{e}_W$. So, if $\tilde{a} \in \text{Im}(\nabla\varphi)(\mathfrak{z})$, then by Corollary 2.9, we can find \tilde{b} making the right triangle in (27) commute. Taking the image under $\Theta_{\mathfrak{z}, X}^{-1}$ yields the commutative triangle on the left, showing that $\Theta^{-1}(\tilde{a}) \in \text{Im}(\varphi)(\eta(\mathfrak{z}))$, and hence that $\tilde{a} \in (\nabla_\eta \text{Im}(\varphi))(\mathfrak{z})$. The converse holds for the same reason, by going this time from left to right.

It then follows from Theorem 7.5 that the adjoint of a sub-schemic motif is again sub-schemic, in case η is sub-schemic itself. Suppose next that \mathfrak{X} is formal, and, for each V -fat point \mathfrak{w} , let $\mathfrak{Y}_{\mathfrak{w}} \subseteq \mathfrak{X}$ be a sub-schemic approximation with the same \mathfrak{w} -rational points. For each W -fat point \mathfrak{z} , let $\tilde{\mathfrak{Y}}_{\mathfrak{z}}$ be defined as $\nabla_\eta(\mathfrak{Y}_{\eta(\mathfrak{z})})$. By what we just proved, $\tilde{\mathfrak{Y}}_{\mathfrak{z}} \subseteq \nabla_\eta \mathfrak{X}$ is a sub-schemic submotif, and one easily verifies that both sieves have the same \mathfrak{z} -rational points, proving the last assertion for sub-schemic adjunctions. The case of a formal adjunction then follows from Theorem 7.5 and Lemma 7.2. \square

8.2. Proposition. *A formal adjunction ∇_η induces a homomorphism of Grothendieck rings $\nabla_\eta: \mathbf{Gr}(\mathbf{Form}_V) \rightarrow \mathbf{Gr}(\mathbf{Form}_W)$. If ∇_η is (sub-)schemic, then this also induces homomorphisms of the corresponding (sub-)schemic Grothendieck rings.*

Proof. By Lemma 8.1, adjunction preserves motivic sites insofar it is (sub-)schemic or formal. As it is compatible with unions and intersections, it preserves scissor relations, and as it is functorial, it preserves isomorphisms as well as products. \square

Before we describe some important instances in which we have adjunction, with applications discussed in §§9 and 11, we give an example of a formal adjunction.

8.3. Example. Given a fat point \mathfrak{z} over an algebraically closed field κ , and $r \geq 2$, let $\Upsilon(\mathfrak{z}) := \Upsilon_r(\mathfrak{z})$ be the fat point with coordinate ring $\kappa + \mathfrak{m}^r \subseteq R$, where (R, \mathfrak{m}) is the Artinian local ring corresponding to \mathfrak{z} . Note that we have a dominant morphism $\mathfrak{z} \rightarrow \Upsilon(\mathfrak{z})$. For simplicity, let us take $r = 2$. For fixed n , let $\mathfrak{l} := \mathfrak{l}_n$ be the n -th jet of a point on a line, with coordinate ring $S := \kappa[\xi]/\xi^n$. For each l , let \mathfrak{w}_l be the fat point in \mathbb{A}_κ^{2l} with ideal of definition generated by all ξ_i^l and Q^n , where

$$Q := \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{2l-1}\xi_{2l}.$$

Let \mathfrak{Y}_l be the image sieve of the morphism $\varphi_l: \mathfrak{w}_l \rightarrow \mathfrak{l}$ induced by $\xi \mapsto Q$. I claim that $\nabla_\Upsilon \mathfrak{l}$ is approximated by the \mathfrak{Y}_l . To this end, fix a fat point \mathfrak{z} with coordinate ring (R, \mathfrak{m}) and let l be its length. A $\Upsilon(\mathfrak{z})$ -rational point $a \in \mathfrak{l}(\Upsilon(\mathfrak{z}))$ is completely determined by the image, denoted again a , of ξ in $\kappa + \mathfrak{m}^2$. Since $a^n = 0$, we must in particular have $a \in \mathfrak{m}^2$ (note that \mathfrak{m}^2 is the maximal ideal of $\Upsilon(\mathfrak{z})$), and hence can be written as $a = b_1b_2 + \cdots + b_{2l-1}b_{2l}$, for some $b_i \in \mathfrak{m}$. Since $b_i^l = 0$ and $a = Q(b_1, \dots, b_{2l})$, the assignment $\xi_i \mapsto b_i$ induces a morphism $\mathfrak{z} \rightarrow \mathfrak{w}_l$ which factors through φ_l . In other words, $a \in \mathfrak{Y}_l(\mathfrak{z})$. Conversely, since Q is quadratic, any \mathfrak{z} -rational point factoring through φ_l must extend to $\Upsilon(\mathfrak{z})$.

Presumably, this argument should extend to any fat point other than \mathfrak{l} and any power $r \geq 2$. To extend this to higher dimensional schemes, we face the problem that a rational point can be given by non-units. This forces us to be able to single out the field elements inside an Artinian local ring R . In characteristic p , this can be done: the elements of $\kappa \subseteq R$ are precisely the p^l -th powers. Using this, a slight modification of the above argument, then yields $\nabla_\Upsilon \mathbb{A}_\kappa^1$ as a formal motif: in the above, replace \mathfrak{w}_l by $\mathbb{A}_{\mathfrak{w}_l}^1$ and \mathfrak{Y}_l by the image of the morphism $\mathbb{A}_{\mathfrak{w}_l}^1 \rightarrow \mathbb{A}_\kappa^1$ given by $\xi \mapsto \xi_0^{p^l} + Q$. It seems likely that we can again extend this argument to arbitrary schemes and arbitrary $r \geq 2$.

8.4. Augmentation. Let $f: W \rightarrow V$ be a morphism of Noetherian Jacobson schemes. Via f , any W -scheme Y becomes a V -scheme, and to make a notational distinction between these two scheme structures, we denote the latter by f_*Y . We will show that f_* constitutes a left adjoint, where the corresponding right adjoint is given by base change: given a V -scheme X , we set

$$f^*X := W \times_V X.$$

8.5. Theorem. *If $f: W \rightarrow V$ is a morphism of finite type of Noetherian Jacobson schemes, then f_* is a functor from \mathbf{Fat}_W to \mathbf{Fat}_V , and as such, it is the left adjoint of f^* . The corresponding adjunction associates to a V -sieve \mathfrak{X} on a V -scheme X , the W -sieve $\nabla_{f_*} \mathfrak{X}$ on f^*X , inducing natural ring homomorphisms on the respective Grothendieck rings, namely $\nabla_{f_*}: \mathbf{Gr}(\mathbf{Sch}_V) \rightarrow \mathbf{Gr}(\mathbf{Sch}_W)$, $\nabla_{f_*}: \mathbf{Gr}(\mathbf{subSch}_V) \rightarrow \mathbf{Gr}(\mathbf{subSch}_W)$, and $\nabla_{f_*}: \mathbf{Gr}(\mathbf{Form}_V) \rightarrow \mathbf{Gr}(\mathbf{Form}_W)$.*

Proof. Let \mathfrak{z} be a W -fat point and let y be its center, that is to say, the closed point on W given as the image under the structure morphism $\mathfrak{z} \rightarrow W$. By the generalized Nullstellensatz ([5, Theorem 4.19]), the image $x := f(y)$ is a closed point on V , and the residue field extension $\kappa(x) \subseteq \kappa(y)$ is finite. As $\kappa(y) \subseteq R/\mathfrak{m}$ is also finite, $f_*\mathfrak{z}$ is a V -fat point, proving the first assertion. The adjunction of f_* and f^* are well-known (and, in any case, easily checked; see, for instance [7, Chapter II.5], but note that left and right are switched there since they are formulated in the dual category of sheaves), proving that $\nabla_{f_*} X = f^* X$. The last statement follows from Proposition 8.2. \square

8.6. Remark. Although $f_*: \mathbf{Fat}_W \rightarrow \mathbf{Fat}_V$ is an embedding of categories, it is, however, not full: so are the closed subschemes in \mathbb{A}_κ^2 defined by the ideals (x^2, y^3) and (x^3, y^2) isomorphic as fat κ -points, but not as fat $\kappa[x]$ -points. Nonetheless, \mathbf{Fat}_W is cofinal in \mathbf{Fat}_V , or, in the terminology of §12 below, both have the same universal point.

8.7. Diminution. Let $f: W \rightarrow V$ be a finite and faithfully flat morphism of Noetherian Jacobson schemes. As opposed to the previous section, we will now consider f^* as a left adjoint. For technical reasons (see Remark 8.10 below for how to circumvent these), we make the following additional assumptions:

- (†) V is of finite type over an algebraically closed field κ and f induces an isomorphism on the underlying varieties.

The second condition implies that for any closed point $x \in V$ there is a unique closed point $y \in W$ lying above it, and hence the closed fiber $f^{-1}(x)$ is a local scheme. Under these assumptions, the base change $f^*\mathfrak{z}$ of a V -fat point \mathfrak{z} is a W -fat point. Indeed, since the problem is local, we may assume $V = \mathrm{Spec} \lambda$ and $W = \mathrm{Spec} \mu$ are affine. Let (R, \mathfrak{m}) be the coordinate ring of \mathfrak{z} , and let $\mathfrak{p} := \lambda \cap \mathfrak{m}$ be the induced maximal ideal of λ . The coordinate ring of $f^*\mathfrak{z}$ is then $S := R \otimes_\lambda \mu$. By base change, S is finite (and flat) over R , whence in particular Artinian. By base change, S is also finite as a μ -module. Since \mathfrak{m} is nilpotent, any maximal ideal of S must contain $\mathfrak{m}S$. Since $S/\mathfrak{m}S = R/\mathfrak{m} \otimes_{\lambda/\mathfrak{p}} \mu/\mathfrak{p}\mu$, since $\lambda/\mathfrak{p} = \kappa$ by the Nullstellensatz, and since R/\mathfrak{m} is a finite extension of the latter, whence trivial, $S/\mathfrak{m}S = \mu/\mathfrak{p}\mu$ is local by assumption (†), showing that S itself is an Artinian local ring, thus proving the claim.

8.8. Theorem. *If $f: W \rightarrow V$ is a finite and faithfully flat morphism satisfying (†), then f^* is the left adjoint of a spheric adjunction, inducing on the Grothendieck rings natural ring homomorphisms $\nabla_{f^*}: \mathbf{Gr}(\mathbf{Sch}_W) \rightarrow \mathbf{Gr}(\mathbf{Sch}_V)$, $\nabla_{f^*}: \mathbf{Gr}(\mathbf{subSch}_W) \rightarrow \mathbf{Gr}(\mathbf{subSch}_V)$, and $\nabla_{f^*}: \mathbf{Gr}(\mathbf{Form}_W) \rightarrow \mathbf{Gr}(\mathbf{Form}_V)$.*

More precisely, for any W -scheme Y , there exists a V -scheme $\nabla_{f^} Y$ and a canonical morphism $\rho_Y: f^*(\nabla_{f^*} Y) \rightarrow Y$ of W -schemes, such that, for any V -fat point \mathfrak{z} , the map sending a \mathfrak{z} -rational point $a: \mathfrak{z} \rightarrow Y$ to the $f^*\mathfrak{z}$ -rational point $\rho_Y \circ f^*a: f^*\mathfrak{z} \rightarrow Y$, induces an isomorphism $(\nabla_{f^*} Y)(\mathfrak{z}) = Y(f^*\mathfrak{z})$.*

If $Z \subseteq Y$ is a closed immersion, then so is the canonical morphism $\nabla_{f^} Z \rightarrow \nabla_{f^*} Y$.*

Proof. Since f is finite and flat, W is locally free over V . Since we may construct each $\nabla_{f^*} Y$ locally and then, by the uniqueness of the universal property of adjoints, glue the pieces together, we may assume that $Y = \mathrm{Spec} B$, $V = \mathrm{Spec} \lambda$, and $W = \mathrm{Spec} \mu$ are affine, and that μ is free over λ (in all applications, we will already have global freeness anyway). Let $\alpha_1, \dots, \alpha_l$ be a basis of μ over λ . Write $B := \mu[x]/(h_1, \dots, h_s)\mu[x]$, for some polynomials h_i over μ . Let \tilde{x} be a new tuple of variables and define a generic tuple of arcs

$$\tilde{x} := \tilde{x}_1\alpha_1 + \dots + \tilde{x}_l\alpha_l$$

in $\mu[\tilde{x}]$. Given any $g \in \mu[x]$, let $\tilde{g}_j \in \lambda[\tilde{x}]$ be defined by the expansion

$$(28) \quad g(\hat{x}) = \sum_{j=1}^l \tilde{g}_j(\tilde{x})\alpha_i.$$

Applying (28) to the h_i , we get polynomials \tilde{h}_{ij} over λ and we let A be the residue ring of $\lambda[\tilde{x}]$ modulo the ideal generated by all these \tilde{h}_{ij} . I claim that $X := \text{Spec } A$ represents $\nabla_{f*} Y$. It follows from (28) that the map $x \mapsto \hat{x}$ yields a μ -algebra homomorphism $B \rightarrow f^* A$, where $f^* A := A \otimes_{\lambda} \mu$ is the base change, and hence a μ -morphism $\rho_Y: f^* X \rightarrow Y$. Fix a λ -fat point \mathfrak{z} , and a \mathfrak{z} -rational point $a: \mathfrak{z} \rightarrow X$. By base change, we get a μ -algebra homomorphism $f^* \mathfrak{z} \rightarrow f^* X$ which composed with ρ_Y induces a $f^* \mathfrak{z}$ -point $\Theta(a): f^* \mathfrak{z} \rightarrow Y$. To prove that the map $a \mapsto \Theta(a)$ establishes an adjunction isomorphism, we construct its converse. Given an $f^* \mathfrak{z}$ -rational point $b: f^* \mathfrak{z} \rightarrow Y$, let $B \rightarrow R \otimes_{\lambda} \mu$ be the corresponding μ -algebra homomorphism, where R is the coordinate ring of \mathfrak{z} . The latter homomorphism is uniquely determined by a tuple \mathbf{u} in $R \otimes_{\lambda} \mu$ such that all $h_i(\mathbf{u}) = 0$. Expanding this tuple as

$$(29) \quad \mathbf{u} = \tilde{\mathbf{u}}_1 \alpha_1 + \cdots + \tilde{\mathbf{u}}_l \alpha_l$$

yields a (unique) tuple $\tilde{\mathbf{u}} := (\tilde{\mathbf{u}}_1, \dots, \tilde{\mathbf{u}}_l)$ over R such that all $\tilde{h}_{ij}(\tilde{\mathbf{u}}) = 0$, determining, therefore, a λ -algebra homomorphism $A \rightarrow R$, whence a λ -morphism $\Lambda(b): \mathfrak{z} \rightarrow X$. So remains to verify that Λ and Θ are mutual inverses. Starting with the $f^* \mathfrak{z}$ -rational point b , we get the \mathfrak{z} -rational point $\Lambda(b)$, which in turn induces the $f^* \mathfrak{z}$ -rational point $\Theta(\Lambda(b))$, given as the composition $\rho_Y \circ f^* \Lambda(b)$. The latter corresponds by (29) to the μ -algebra homomorphism $B \rightarrow f^* A \rightarrow f^* R$ given by $x \mapsto \hat{x} \mapsto \mathbf{u}$, showing that $\Theta(\Lambda(b)) = b$. If, on the other hand, we start with the \mathfrak{z} -rational point a , given by $\hat{x} \mapsto \tilde{\mathbf{u}}$, we get the $f^* \mathfrak{z}$ -rational point $\Theta(a)$, given by $x \mapsto \mathbf{u}$, where \mathbf{u} is as in (29). Hence $\Lambda(\Theta(a))$ is given by $\hat{x} \mapsto \tilde{\mathbf{u}}$, that is to say, is equal to a , as we needed to show.

To prove the last assertion, assume that Z is a closed subscheme of Y , so that its coordinate ring is of the form $B/(h_{s+1}, \dots, h_t)B$ for some additional polynomials $h_i \in \mu[x]$. Hence $\nabla_{f*} Z$ is the closed subscheme of $\nabla_{f*} Y$ given by the \tilde{h}_{ij} for $s < i \leq t$. \square

Immediately from the above proof, by taking $Y = f^* X$, we have the following result, which we will use in the next section:

8.9. Corollary. *If $f: W \rightarrow V$ is a finite and faithfully flat morphism satisfying (\dagger) , then we have for each V -scheme X , a canonical V -morphism $\rho_X: \nabla_{f*} f^* X \rightarrow X$. If $Z \subseteq X$ is a closed immersion, then so is the canonical morphism $\nabla_{f*} f^* Z \rightarrow \nabla_{f*} f^* X$. \square*

8.10. Remark. Without assumption (\dagger) , the pull-back of a V -fat point \mathfrak{z} is only a zero-dimensional W -scheme, and hence a disjoint sum of W -fat points $f^* \mathfrak{z} = \mathfrak{w}_1 \sqcup \cdots \sqcup \mathfrak{w}_s$. We can then still make sense of $Y(f^* \mathfrak{z})$, as the disjoint union $Y(\mathfrak{w}_1) \sqcup \cdots \sqcup Y(\mathfrak{w}_s)$, and the adjunction condition then becomes that this must be equal to $(\nabla_{f*} Y)(\mathfrak{z})$. Since nowhere in the above proof we used that $f^* R$ is local, we therefore can omit condition (\dagger) from the statement of Theorem 8.8.

We have the following commutation rule for adjunctions in a Cartesian square:

8.11. Theorem (Projection Formula). *Let $f: W \rightarrow V$ be a finite and faithfully flat morphism of Noetherian Jacobson schemes satisfying (\dagger) , let $u: \tilde{V} \rightarrow V$ be a morphism of*

finite type, and let

$$(30) \quad \begin{array}{ccc} \tilde{W} & \xrightarrow{\tilde{f}} & \tilde{V} \\ \tilde{u} \downarrow & & \downarrow u \\ W & \xrightarrow{f} & V \end{array}$$

be the base change diagram, where $\tilde{W} := W \times_V \tilde{V}$. We have an identity of adjunctions

$$\nabla_{\tilde{f}*} \nabla_{\tilde{u}*} = \nabla_{u*} \nabla_{f*}$$

from W -sieves to \tilde{V} -sieves.

Proof. Note that \tilde{f} is again finite and faithfully flat, satisfying (\dagger) , so that the diminution $\nabla_{\tilde{f}*}$ makes sense. To prove the identity we have to check it on each W -sieve \mathfrak{Y} and each \tilde{V} -fat point \mathfrak{z} , becoming

$$(\nabla_{\tilde{f}*} \nabla_{\tilde{u}*} \mathfrak{Y})(\mathfrak{z}) = \mathfrak{Y}(\tilde{u}_* \tilde{f}^* \mathfrak{z}) \stackrel{?}{=} \mathfrak{Y}(f^* u_* \mathfrak{z}) = (\nabla_{u*} \nabla_{f*} \mathfrak{Y})(\mathfrak{z}).$$

But one easily verifies that we have an equality of W -fat points

$$\tilde{u}_* \tilde{f}^* \mathfrak{z} = f^* u_* \mathfrak{z}$$

concluding the proof of the theorem. \square

8.12. Frobenius transform. Assume for the remainder of this section that the base ring is a field κ of characteristic $p > 0$. Let us denote the Frobenius homomorphism $a \mapsto a^p$ on a κ -algebra A by \mathbf{F} , or in case we need to specify the ring by \mathbf{F}_A , so that we have in particular a commutative diagram

$$(31) \quad \begin{array}{ccc} \kappa & \xrightarrow{\mathbf{F}_\kappa} & \kappa \\ \downarrow & & \downarrow \\ A & \xrightarrow{\mathbf{F}_A} & A. \end{array}$$

Due to the functorial nature, we can glue these together and hence obtain on any κ -scheme X a corresponding endomorphism \mathbf{F}_X .

Diagram (31) implies that \mathbf{F}_A is not a κ -algebra homomorphism. To overcome this difficulty, we assume κ is perfect, so that \mathbf{F} is an isomorphism on κ . To make (31) into a κ -algebra homomorphism, we must view the second copy of A with a different κ -algebra structure, namely, the one inherited from the composite homomorphism $\kappa \xrightarrow{\mathbf{F}} \kappa \rightarrow A$. Several notational devices have been proposed (see for instance [7, Chapter IV, Remark 2.4.1] or [17, Chapter 8.1.c]), but we will use the one already introduced in the previous section: the push-forward of A along \mathbf{F} will be denoted $\mathbf{F}_* A$. In other words, $\mathbf{F}_* A$ is A with its κ -action given by $u \cdot a = u^p a$. Since κ is perfect, $A \cong \mathbf{F}_* A$ as rings, and in many instances, even as κ -algebras. In particular, (31) yields a κ -algebra homomorphism

$A \xrightarrow{\mathbf{F}} \mathbf{F}_* A$, called the κ -linear Frobenius. The image of the κ -linear Frobenius homomorphism $A \xrightarrow{\mathbf{F}} \mathbf{F}_* A$ is the subring of A consisting of all p -th powers, and we will simply denote it by $\mathbf{F}A$ (rather than the more common A^p , which might lead to confusions with Cartesian powers). Hence, pushing forward the inclusion homomorphism $\mathbf{F}A \subseteq A$ gives a factorization of the κ -algebra homomorphism \mathbf{F} as $A \rightarrow \mathbf{F}_* \mathbf{F}A \subseteq \mathbf{F}_* A$, where the first homomorphism is an isomorphism if and only if A is reduced. For instance, if $A = \kappa[x]$, then $\mathbf{F}A = \kappa[x^p]$, so that this factorization is given by the sequence of κ -algebra homomorphisms

$$(32) \quad \begin{array}{ccccc} \kappa[x] & \xrightarrow{\cong} & \mathbf{F}_* \kappa[x^p] & \subseteq & \mathbf{F}_* \kappa[x] & \xrightarrow{\cong} & \kappa[x] \\ & & & & h \mapsto \sigma & & \tilde{h} \\ & & g \mapsto & g^p \mapsto & g(x^p), & & \end{array}$$

where \tilde{h} is obtained from h by replacing each coefficient with its (unique) p -th root. So, from this we can calculate $\mathbf{F}_* A$ for A of the form $\kappa[x]/(f_1, \dots, f_s)\kappa[x]$ as

$$\mathbf{F}_* A \cong \kappa[x]/(\tilde{f}_1, \dots, \tilde{f}_s)\kappa[x],$$

with $\tilde{f}_i = \sigma(f_i)$ as in (32). The κ -linear Frobenius $A \rightarrow \mathbf{F}_* A$ is then the induced homomorphism by the composite map $g \mapsto g(x^p)$ from (32).

Similarly, viewing X as a κ -scheme via the composition $X \rightarrow \mathrm{Spec} \kappa \xrightarrow{\mathbf{F}} \mathrm{Spec} \kappa$, it will be denoted by $\mathbf{F}_* X$, yielding a morphism of κ -schemes $\mathbf{F}_X: \mathbf{F}_* X \rightarrow X$, called the κ -linear Frobenius. Its scheme-theoretic closure will be denoted by $\mathbf{F}X$, so that we have a dominant morphism $X \rightarrow \mathbf{F}X$, yielding a factorization

$$(33) \quad \mathbf{F}_X: \mathbf{F}_* X \rightarrow \mathbf{F}_* \mathbf{F}X \subseteq X$$

of \mathbf{F}_X , where the closed immersion $\mathbf{F}_* \mathbf{F}X \subseteq X$ is the identity if and only if X is a variety. In particular, $\mathbf{F}X$ is the Zariski closure of $\mathrm{Im}(\mathbf{F}_X)$ in X .

We could view \mathbf{F}_κ as an automorphism of the base to get by Theorem 8.5 an adjunction pair $(\mathbf{F}_\kappa^*, \mathbf{F}_{\kappa*})$. However, since X and $\mathbf{F}_{\kappa*} \mathbf{F}_\kappa^* X$ are isomorphic as κ -schemes, this merely induces an action of the Frobenius. For the same reason, diminution does not induce any interesting endomorphism on the Grothendieck ring. Instead we take a relative point of view. To a morphism $\varphi: Y \rightarrow X$ of κ -schemes, we can associate two commutative squares; the base change and the Frobenius square. Combined into a single commutative diagram of κ -morphisms, we have

$$\begin{array}{ccccc} \mathbf{F}_* Y & & \mathbf{F}_Y & & \\ \downarrow \mathbf{F}_* \varphi & \searrow & \downarrow & \searrow & \\ & \mathbf{F}_X^* Y & \xrightarrow{\mathbf{F}_X \times \mathbf{1}_Y} & Y & \\ & \downarrow & & \downarrow \varphi & \\ & \mathbf{F}_* X & \xrightarrow{\mathbf{F}_X} & X & \end{array}$$

where $\mathbf{F}_X^* Y := \mathbf{F}_* X \times_X Y$ is the pull-back of Y along \mathbf{F}_X , called the *Frobenius transform of Y in X* , and where the canonical projection $\mathbf{F}_X \times \mathbf{1}_Y: \mathbf{F}_X^* Y = \mathbf{F}_* X \times_X Y \rightarrow Y$ is called the *relative Frobenius on Y over X* . In case φ is a closed immersion, the natural morphism $\mathbf{F}_* Y \rightarrow \mathbf{F}_X^* Y$ is then also a closed immersion. We can calculate it explicitly

in case $X = \operatorname{Spec} A$ is affine and Y is defined by the ideal $I \subseteq A$. Traditionally, one denotes the ideal generated by the image of I under the Frobenius \mathbf{F}_A by $I^{[p]}$; it is the ideal generated by all f^p with $f \in I$. With this notation, we have

$$\mathbf{F}_X^* Y = \mathbf{F}_*(\operatorname{Spec} A/I^{[p]}).$$

In particular, applying σ from (32) to the previous isomorphism in case X is affine space, we get:

8.13. Corollary. *If Y is the closed subscheme of \mathbb{A}_κ^n with ideal of definition (f_1, \dots, f_s) , then $\mathbf{F}_{\mathbb{A}_\kappa^n}^* Y$ is the closed subscheme of \mathbb{A}_κ^n with ideal of definition $(f_1(x^p), \dots, f_s(x^p))$, and the relative Frobenius $\mathbf{F}_{\mathbb{A}_\kappa^n} \times \mathbf{1}_Y$ is the map induced by $x \mapsto x^p$. \square*

The assignment $\mathfrak{z} \mapsto \mathbf{F}\mathfrak{z}$ constitutes a functor on $\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa$, which will play the role of left adjoint. However, in this case, the adjunction will only be sub-schemic, via the following right adjoint. For each κ -scheme Y , we define a sub-schemic motif \mathfrak{F}_Y , called its *Frobenius motif*. In order to do this, we will work locally: show that it is a right adjoint locally, and then deduce its uniqueness and existence, as well as right adjointness, globally. So let Y be affine, say, a closed subscheme of \mathbb{A}_κ^n , and let $\mathbf{F}_{\mathbb{A}_\kappa^n} \times \mathbf{1}_Y : \mathbf{F}_{\mathbb{A}_\kappa^n}^* Y \rightarrow Y$ be the corresponding relative Frobenius. Set $\mathfrak{F}_Y := \operatorname{Im}(\mathbf{F}_{\mathbb{A}_\kappa^n} \times \mathbf{1}_Y)$, so that it is a sub-schemic motif on Y . To see that this is independent from the choice of closed immersion, we prove the adjunction formula

$$(34) \quad Y(\mathbf{F}\mathfrak{z}) = \mathfrak{F}_Y(\mathfrak{z})$$

for any fat point \mathfrak{z} . More precisely, the canonical (dominant) morphism $\mathfrak{z} \rightarrow \mathbf{F}\mathfrak{z}$ induces a map $Y(\mathbf{F}\mathfrak{z}) \rightarrow Y(\mathfrak{z})$. By Lemma 2.3 it is injective, and we want to show that its image is $\mathfrak{F}_Y(\mathfrak{z})$. Let $(f_1, \dots, f_s)\kappa[x]$ be the ideal defining Y . By Corollary 8.13, the Frobenius transform $\mathbf{F}_{\mathbb{A}_\kappa^n}^* Y$ is given by the ideal $(f_1(x^p), \dots, f_s(x^p))\kappa[x]$. An $\mathbf{F}\mathfrak{z}$ -rational point a in Y corresponds to a κ -algebra homomorphism $A \rightarrow \mathbf{F}R$, where R is the coordinate ring of \mathfrak{z} , and hence to a solution of $f_1 = \dots = f_s = 0$ in R of the form \mathbf{r}^p . The image a' of a in $Y(\mathfrak{z})$ corresponds to the composition $A \rightarrow \mathbf{F}R \subseteq R$. Since \mathbf{r} is a solution in R of the equations defining $\mathbf{F}_{\mathbb{A}_\kappa^n}^* Y$, it induces a \mathfrak{z} -rational point $b : \mathfrak{z} \rightarrow \mathbf{F}_{\mathbb{A}_\kappa^n}^* Y$ such that $a' = (\mathbf{F}_{\mathbb{A}_\kappa^n} \times \mathbf{1}_Y)(\mathfrak{z})(b)$, proving that $a' \in \mathfrak{F}_Y(\mathfrak{z})$. Conversely, by reversing these arguments, we see that any such \mathfrak{z} -rational point is induced by a p -th power in R , and hence comes from a $\mathbf{F}\mathfrak{z}$ -rational point. This concludes the proof of (34) when Y is affine, and proves in particular that \mathfrak{F}_Y does not depend on the choice of closed immersion. For arbitrary Y , let Y_1, \dots, Y_m be an open affine covering. For each Y_i and each intersection $Y_i \cap Y_j$, we have an equality (34). Hence we may glue all pieces together to obtain a sub-schemic motif \mathfrak{F}_Y satisfying (34). In particular, in view of Proposition 8.2, we proved:

8.14. Theorem. *The functors $\mathfrak{z} \mapsto \mathbf{F}\mathfrak{z}$ and $Y \mapsto \mathfrak{F}_Y$ constitute a sub-schemic adjunction. In particular, we get induced endomorphisms $\nabla_{\mathbf{F}}$ on $\mathbf{Gr}(\operatorname{subSclh}_\kappa)$ and $\mathbf{Gr}(\operatorname{Form}_\kappa)$. \square*

Unraveling the definitions, the action of this adjunction on a motif \mathfrak{Y} on a scheme Y is given by

$$\nabla_{\mathbf{F}}\mathfrak{Y} = \mathfrak{Y} \cap \mathfrak{F}_Y.$$

Moreover, if $Y = \mathbb{A}_\kappa^n$, then \mathfrak{F}_Y is just $\operatorname{Im}(\mathbf{F}_{\mathbb{A}_\kappa^n})$, the image of the κ -linear Frobenius. Therefore, if a motif \mathfrak{Y} has an ambient space which is affine, we may take it to be an affine space \mathbb{A}_κ^n , so that

$$\nabla_{\mathbf{F}}\mathfrak{Y} = \mathfrak{Y} \cap \operatorname{Im}(\mathbf{F}_{\mathbb{A}_\kappa^n}).$$

8.15. Families of motives. Let $s: \mathfrak{Y} \rightarrow \mathfrak{X}$ be an algebraic morphism of V -sieves (see Remark 8.17 below for the non-algebraic case). Hence, we can find ambient spaces Y and X of \mathfrak{Y} and \mathfrak{X} respectively, and a morphism $\varphi: Y \rightarrow X$ of V -schemes extending s . We explain now how we may view s as a family of V -sieves, by associating to each V -morphism $a: V \rightarrow X$, a V -sieve \mathfrak{Y}_a as follows. We may view \mathfrak{Y} as an X -sieve via φ , and, in accordance with previous notation, we denote this X -sieve by $\varphi_*\mathfrak{Y}$. Using a to as augmentation map, we define

$$\mathfrak{Y}_a := \nabla_{a*} \varphi_* \mathfrak{Y},$$

called the *specialization of \mathfrak{Y} at a* . By Theorem 8.5, this is a sieve on the base change $\nabla_{a*} Y = a^*Y = V \times_X Y$. To see that the specialization \mathfrak{Y}_a is independent from the choice of ambient space Y , we simply observe that

$$(35) \quad \mathfrak{Y}_a(\mathfrak{z}) = V(\mathfrak{z}) \times_{\mathfrak{X}(\mathfrak{z})} \mathfrak{Y}(\mathfrak{z}) = \{(r, b) \in V(\mathfrak{z}) \times \mathfrak{Y}(\mathfrak{z}) \mid a(\mathfrak{z})(r) = s(\mathfrak{z})(b)\}$$

as a subset of $V(\mathfrak{z}) \times_{X(\mathfrak{z})} Y(\mathfrak{z})$, for any V -fat point \mathfrak{z} . Immediately from Theorem 8.5, we have:

8.16. Proposition. *The specialization of a schemic, sub-schemic, or formal motif is again of the same type.* \square

8.17. Remark. We can even apply this theory to a non-algebraic morphism $s: \mathfrak{Y} \rightarrow \mathfrak{X}$ of formal motives. Indeed, let $\mathfrak{Z}_{\mathfrak{z}} \subseteq \mathfrak{Y}$ be sub-schemic approximations of \mathfrak{Y} . By Theorem 3.7, the restriction $s|_{\mathfrak{Z}_{\mathfrak{z}}}$ is algebraic. Hence, given $a: V \rightarrow X$, the specialization $(\mathfrak{Z}_{\mathfrak{z}})_a$ is sub-schemic by Proposition 8.16. Using (35), it is not hard to show that these specializations $(\mathfrak{Z}_{\mathfrak{z}})_a$ are approximations of \mathfrak{Y}_a (as defined by the right hand side of (35)), showing that the latter is formal too.

9. ARC SCHEMES

From now on, our base scheme will (almost always) be an algebraically closed field κ . Fix a fat point \mathfrak{z} and let $j: \mathfrak{z} \rightarrow \text{Spec } \kappa$ be its structure morphism. Clearly, it is flat and finite and satisfies condition (†), and so both augmentation and diminution with respect to j are well-defined. We define the *arc functor* of \mathfrak{z} , as a double adjunction⁷

$$\nabla_{\mathfrak{z}} := \nabla_{j*} \circ \nabla_{j*}.$$

In other words, given a motif \mathfrak{X} on a κ -scheme X , and a fat point \mathfrak{w} , we have

$$(\nabla_{\mathfrak{z}})(\mathfrak{w}) = \mathfrak{X}(j_* j^* \mathfrak{w})$$

where $j_* j^* \mathfrak{w}$ is the product $\mathfrak{z} \times_{\kappa} \mathfrak{w}$ viewed as a fat point over κ , denoted henceforth simply by $\mathfrak{z}\mathfrak{w}$. Applied to a κ -scheme X , we get the so-called *arc scheme* $\nabla_{\mathfrak{z}} X$, whose \mathfrak{w} -rational points are in one-one correspondence with the $\mathfrak{z}\mathfrak{w}$ -rational points of X , and so we will identify henceforth

$$X(\mathfrak{z}\mathfrak{w}) = (\nabla_{\mathfrak{z}} X)(\mathfrak{w}).$$

Moreover, we have by Corollary 8.9 a canonical morphism

$$(36) \quad \rho_X: \nabla_{\mathfrak{z}} X \rightarrow X.$$

9.1. Remark. In other words, $\nabla_{\mathfrak{z}} X$ is the Hilbert scheme classifying all maps from \mathfrak{z} to X . When $\mathfrak{z} = \text{Spec}(\kappa[\xi]/\xi^n \kappa[\xi])$, the resulting arc scheme is also known in the literature as a *jet scheme* (though we prefer to reserve the latter nomenclature for schemes of the form $J_Y^n X$).

⁷See §11 below for the corresponding single adjunction.

9.2. Remark. By the argument in the proof of Theorem 8.8, for any fat point \mathfrak{z} , we may choose a basis $\Delta = \{\alpha_0, \dots, \alpha_{l-1}\}$ of its coordinate ring (R, \mathfrak{m}) with some additional properties. In particular, unless noted explicitly, we will always assume that the first base element is 1 and that the remaining ones belong to \mathfrak{m} . Moreover, once the basis is fixed, we let \tilde{x} be the l -tuple of arc variables $(\tilde{x}_0, \dots, \tilde{x}_{l-1})$, so that $\hat{x} = \tilde{x}_0 + \tilde{x}_1\alpha_1 + \dots + \tilde{x}_{l-1}\alpha_{l-1}$ is the corresponding generic arc. It follows from (28) that $\tilde{f}_0 = f(\tilde{x}_0)$, for any $f \in \kappa[x]$. By [13, §2.1], we may choose Δ so that, with $\mathfrak{a}_i := (\alpha_i, \dots, \alpha_{l-1})R$, we have a Jordan-Holder composition series⁸

$$\mathfrak{a}_l = 0 \subsetneq \mathfrak{a}_{l-1} \subsetneq \mathfrak{a}_{l-2} \subsetneq \dots \subsetneq \mathfrak{a}_1 = \mathfrak{m} \subsetneq \mathfrak{a}_0 = R.$$

Given $r \in R$, we expand it as in (29) in the basis as $r = r_0 + r_1\alpha_1 + \dots + r_{l-1}\alpha_{l-1}$, with $r_i \in \kappa$. I claim that $r_j = 0$ for $j < i$ whenever $r \in \mathfrak{a}_i$. Indeed, if not, let $j < i$ be minimal so that there exists a counterexample with $r_j \neq 0$. By minimality, $r = r_j\alpha_j + r_{j+1}\alpha_{j+1} + \dots \in \mathfrak{a}_i$ showing that $\alpha_j \in \mathfrak{a}_{j+1}$, since r_j is invertible. However, this implies that $\mathfrak{a}_j = \mathfrak{a}_{j+1}$, contradiction. From this, it is now easy to see that the first m basis elements of Δ form a basis of $R_m := R/\mathfrak{a}_{m+1}$. Therefore, calculating \tilde{f}_m in (28) for $f \in \kappa[x]$ does not depend on whether we work over R or with R_m , and hence, in particular, $\tilde{f}_m \in \kappa[\tilde{x}_0, \dots, \tilde{x}_m]$ for every $m < l$.

With these observations, we can now prove the following important openness property of arcs:

9.3. Theorem. *Given a κ -scheme X , a fat point \mathfrak{z} , and an open $U \subseteq X$, we have isomorphisms*

$$(37) \quad \nabla_{\mathfrak{z}} U \cong \rho_X^{-1}(U) \cong \nabla_{\mathfrak{z}} X \times_X U.$$

Proof. By the universal property of adjunction, whence of arcs, it suffices to verify (37) in case $X = \text{Spec } B \subseteq \mathbb{A}_{\kappa}^n$ is affine and $U = \text{Spec}(B_f)$ is a basic open subset. Let A be the coordinate ring of $\nabla_{\mathfrak{z}} X$. Since U is the closed subscheme of \mathbb{A}_B^1 given by $g := fy - 1 = 0$, the corresponding arc scheme $\nabla_{\mathfrak{z}} U$ is the closed subscheme of \mathbb{A}_A^n with coordinate ring $A' := A[\tilde{y}]/(\tilde{g}_0, \dots, \tilde{g}_{l-1})A[\tilde{y}]$, where l is the length of $\mathfrak{z} = \text{Spec } R$ and the \tilde{g}_i are given by (28), with \tilde{y} a tuple of l variables. By Remark 9.2, we may calculate the \tilde{g}_i using any basis $\alpha_0 = 1, \dots, \alpha_{l-1}$ of R , and so we may assume it has the properties discussed in that remark. In particular, by the last observation in that remark, each \tilde{g}_i only depends on $\tilde{y}_0, \dots, \tilde{y}_i$. Clearly, $\tilde{g}_0 = \tilde{f}_0\tilde{y}_0 - 1$. In particular, the A -subalgebra of A' generated by \tilde{y}_0 is just $A_{\tilde{f}_0}$. We will prove by induction, that each \tilde{y}_i belongs to this subalgebra, and hence $A' = A_{\tilde{f}_0}$, as we needed to prove.

To verify the claim, we may assume by induction that $\tilde{y}_0, \dots, \tilde{y}_{i-1}$ belong to $A_{\tilde{f}_0}$. The coefficient of α_i in the expansion

$$(38) \quad (\tilde{f}_0 + \tilde{f}_1\alpha_1 + \dots + \tilde{f}_{l-1}\alpha_{l-1})(\tilde{y}_0 + \tilde{y}_1\alpha_1 + \dots + \tilde{y}_{l-1}\alpha_{l-1})$$

is equal to \tilde{g}_i , whence zero in A' . As observed in Remark 9.2, the choice of basis allows us to ignore all terms with α_j for $j > i$. Put differently, upon replacing R by $R/(\alpha_{i+1}, \dots, \alpha_{l-1})R$, which does not effect the calculation of \tilde{g}_i , we may assume that they are zero in (38). Hence,

$$\tilde{g}_i = \tilde{f}_0\tilde{y}_i + \text{terms involving only } \tilde{y}_0, \dots, \tilde{y}_{i-1}$$

⁸Writing R as a homomorphic image of $\kappa[y]$ so that $y := (y_1, \dots, y_e)$ generates \mathfrak{m} , let $\mathfrak{a}(\alpha)$, for $\alpha \in \mathbb{Z}_{\geq 0}^n$, be the ideal in R generated by all y^β with β lexicographically larger than α . Then we may take Δ to be all monomials y^α such that $y^\alpha \notin \mathfrak{a}(\alpha)$, ordered lexicographically.

proving the claim, since \tilde{f}_0 is clearly invertible in $A_{\tilde{f}_0}$. \square

Before we proceed, some simple examples are in order. It is clear from the definitions that $\nabla_{\mathfrak{z}} \mathbb{A}_\kappa^1 = \mathbb{A}_\kappa^l$, where l is the length of \mathfrak{z} . We will generalize this in Theorem 9.10 below. Arc spaces are sensitive to singularities, as the next examples show:

9.4. Example. Let us calculate the arc scheme of the cusp C given by the equation $x^2 - y^3 = 0$ along the fat point \mathfrak{z} with coordinate ring the four dimensional algebra $R := \kappa[\xi, \zeta]/(\xi^2, \zeta^2)\kappa[\xi, \zeta]$, using the basis $\Delta := \{1, \xi, \zeta, \xi\zeta\}$ (in the order listed), and corresponding arc variables $\tilde{x} = (\tilde{x}_{00}, \tilde{x}_{10}, \tilde{x}_{01}, \tilde{x}_{11})$ and $\tilde{y} = (\tilde{y}_{00}, \tilde{y}_{10}, \tilde{y}_{01}, \tilde{y}_{11})$. One easily calculates that $\nabla_{\mathfrak{z}} C$ is given by the equations

$$\begin{aligned} \tilde{x}_{00}^2 &= \tilde{y}_{00}^3 \\ 2\tilde{x}_{00}\tilde{x}_{10} &= 3\tilde{y}_{00}^2\tilde{y}_{10} \\ 2\tilde{x}_{00}\tilde{x}_{01} &= 3\tilde{y}_{00}^2\tilde{y}_{01} \\ 2\tilde{x}_{00}\tilde{x}_{11} + 2\tilde{x}_{10}\tilde{x}_{01} &= 3\tilde{y}_{00}^2\tilde{y}_{11} + 6\tilde{y}_{00}\tilde{y}_{10}\tilde{y}_{01}. \end{aligned}$$

Note that above the singular point $\tilde{x}_{00} = 0 = \tilde{y}_{00}$, the fiber consist of two 4-dimensional hyperplanes, whereas above any regular point, it is a 3-dimensional affine space, the expected value by Theorem 9.10 below.

9.5. Example. Another example is classical: let $R_2 = \kappa[\xi]/\xi^2\kappa[\xi]$ be the ring of dual numbers and $\mathfrak{l}_2 := \text{Spec}(R_2)$ the corresponding fat point. Then one verifies that a κ -rational point on $\nabla_{\mathfrak{l}_2} X$ is given by a κ -rational point P on X , and a tangent vector \mathbf{v} to X at P , that is to say, an element in the kernel of the Jacobian matrix $\text{Jac}_X(P)$.

9.6. Example. As a last example, we calculate $\nabla_{\mathfrak{l}_n} \mathfrak{l}_m$, where \mathfrak{l}_n is the n -th jet of the origin on the affine line, that is to say, $\text{Spec}(\kappa[\xi]/\xi^n\kappa[\xi])$. With $\tilde{x} = \tilde{x}_0 + \tilde{x}_1\xi + \dots + \tilde{x}_{n-1}\xi^{n-1}$, we will expand \tilde{x}^m in the basis $\{1, \xi, \dots, \xi^{n-1}\}$ of $\kappa[\xi]/\xi^n\kappa[\xi]$; the coefficients of this expansion then generate the ideal of definition of $\nabla_{\mathfrak{l}_n} \mathfrak{l}_m$. A quick calculation shows that these generators are the polynomials

$$g_s(\tilde{x}_0, \dots, \tilde{x}_{n-1}) := \sum_{i_1 + \dots + i_m = s} \tilde{x}_{i_1} \tilde{x}_{i_2} \dots \tilde{x}_{i_m}$$

for $s = 0, \dots, n-1$, where the i_j run over $\{0, \dots, n-1\}$. Note that $g_0 = \tilde{x}_0^m$. One shows by induction that $(\tilde{x}_0, \dots, \tilde{x}_s)\kappa[\tilde{x}]$ is the unique minimal prime ideal of $\nabla_{\mathfrak{l}_n} \mathfrak{l}_m$, where $s = \lceil \frac{n}{m} \rceil$ is the *round-up* of n/m , that is to say, the least integer greater than or equal to n/m . In particular, $\nabla_{\mathfrak{l}_n} \mathfrak{l}_m$ is irreducible (but not reduced) of dimension $n - \lceil \frac{n}{m} \rceil$.

Immediately from Theorems 8.5 and 8.8, we get:

9.7. Theorem. *For each fat point \mathfrak{z} , the arc functor $\nabla_{\mathfrak{z}}$ induces a ring endomorphism on each of the motivic Grothendieck rings $\mathbf{Gr}(\mathcal{S}\mathbf{ch}_\kappa)$, $\mathbf{Gr}(\mathbf{sub}\mathcal{S}\mathbf{ch}_\kappa)$ and $\mathbf{Gr}(\mathbf{Form}_\kappa)$. \square*

In case of complete formal motives, we even have:

9.8. Lemma. *For any closed immersion $Y \subseteq X$ of κ -schemes, and any fat point \mathfrak{z} , we have isomorphisms*

$$\nabla_{\mathfrak{z}}(\hat{X}_Y) \cong \nabla_{\mathfrak{z}} X \times_X \hat{X}_Y \cong (\widehat{\nabla_{\mathfrak{z}} X})_{\rho^{-1}(Y)},$$

where $\rho: \nabla_{\mathfrak{z}} X \rightarrow X$ is the canonical map from (36).

Proof. Let $U := X - Y$. By Proposition 7.1, we have an equality

$$(39) \quad -\hat{X}_Y^\circ = U^\circ$$

of sieves on X . By Theorem 7.5, we may pull back (39) under the map $\rho: \nabla_3 X \rightarrow X$, to get a relation

$$-(\nabla_3 X \times_X \hat{X}_Y)^\circ = \rho^*(-\hat{X}_Y^\circ) = \rho^*U^\circ = (\nabla_3 X \times_X U)^\circ = (\nabla_3 U)^\circ$$

where we used the openness of arcs (Theorem 9.3) for the last equality. On the other hand, taking arcs in identity (39), yields

$$-\nabla_3 \hat{X}_Y^\circ = \nabla_3(-\hat{X}_Y^\circ) = \nabla_3 U^\circ = (\nabla_3 U)^\circ$$

where one easily checks that arc functors commute with complements of complete sieves. Combining both identities and taking complements then proves the first isomorphism.

To see the second isomorphism, we may assume, in view of the local nature of arcs, that $X = \text{Spec } A$ is affine. Let $I \subseteq A$ be the ideal of definition of Y , so that the global sections of \hat{X}_Y is the completion \hat{A}_I of A with respect to I . Let $A[y]/J$ be the coordinate ring of the arc scheme $\nabla_3 X$, for some $J \subseteq A[y]$ and some tuple of variables y . By the first isomorphism, the global section ring of $\nabla_3 \hat{X}_Y$ is equal to the base change $\hat{A}_I[y]/J\hat{A}_I[y]$. The ideal defining $\rho^{-1}(Y)$ in $\nabla_3 X$ is $I(A[y]/J)$, and the completion of $A[y]/J$ with respect to this ideal is $\hat{A}_I[y]/J\hat{A}_I[y]$, proving the second isomorphism. \square

It is easy to check that $\nabla_v \nabla_w = \nabla_{vw} = \nabla_w \nabla_v$, so that all arc functors commute with one another. If κ has positive characteristic, we also have a Frobenius adjoint acting on the sub-schematic and formal Grothendieck rings, and we have the following commutation relation

$$(40) \quad \nabla_3 \nabla_F = \nabla_F \nabla_{F_3}$$

for any fat point \mathfrak{z} . Indeed, we verify this on an arbitrary motif \mathfrak{X} and a fat point \mathfrak{w} . The left hand side of (40) becomes

$$\begin{aligned} (\nabla_3 \nabla_F)(\mathfrak{X})(\mathfrak{w}) &= \nabla_3(\nabla_F \mathfrak{X})(\mathfrak{w}) \\ &= (\nabla_F \mathfrak{X})(\mathfrak{z}\mathfrak{w}) \\ &= \mathfrak{X}(\mathbf{F}(\mathfrak{z}\mathfrak{w})), \end{aligned}$$

whereas the right hand side becomes

$$\begin{aligned} (\nabla_F \nabla_{F_3})(\mathfrak{X})(\mathfrak{w}) &= \nabla_F(\nabla_{F_3} \mathfrak{X})(\mathfrak{w}) \\ &= (\nabla_{F_3} \mathfrak{X})(\mathbf{F}\mathfrak{w}) \\ &= \mathfrak{X}((\mathbf{F}\mathfrak{z})(\mathbf{F}\mathfrak{w})), \end{aligned}$$

and these are both equal since an easy calculation shows that $\mathbf{F}(\mathfrak{z}\mathfrak{w}) = (\mathbf{F}\mathfrak{z})(\mathbf{F}\mathfrak{w})$.

Arcs and locally trivial fibrations. By adjunction, any morphism $\bar{\mathfrak{z}} \rightarrow \mathfrak{z}$ of fat points induces a natural transformation of arc functors $\nabla_{\bar{\mathfrak{z}}} \rightarrow \nabla_{\mathfrak{z}}$. In particular, taking $\bar{\mathfrak{z}}$ to be the geometric point given by κ itself, we get a canonical morphism $\nabla_{\bar{\mathfrak{z}}} \mathfrak{X} \rightarrow \mathfrak{X}$, for any motif \mathfrak{X} , since ∇_{κ} is the identity functor. In case $\mathfrak{X} = X^\circ$ is representable, this is none other than the canonical morphism $\rho_X: \nabla_3 X \rightarrow X$ from (36). To formulate the key property of this morphism, we need a definition.

We call a morphism $Y \rightarrow X$ of κ -schemes a *locally trivial fibration with fiber Z* if for each (closed) point $P \in X$, we can find an open $U \subseteq X$ containing P such that the restriction of $Y \rightarrow X$ to U is isomorphic with the projection $U \times_\kappa Z \rightarrow U$.

9.9. Lemma. *If $f: Y \rightarrow X$ is a locally trivial fibration of κ -schemes with fiber Z , then $[Y] = [X] \cdot [Z]$ in $\mathbf{Gr}(\mathbf{subSch}_\kappa)$.*

Proof. By definition and compactness, there exists a finite open covering $X = X_1 \cup \dots \cup X_n$, so that

$$f^{-1}(X_i) \cong X_i \times_\kappa Z,$$

for $i = 1, \dots, n$. In fact, for any non-empty subset $I \subseteq \{1, \dots, n\}$, we have an isomorphism $f^{-1}(X_I) \cong X_I \times_\kappa Z$, and hence, after taking classes in $\mathbf{Gr}(\mathbf{subSch}_\kappa)$, we get $[f^{-1}(X_I)] = [X_I] \cdot [Z]$. Since the $f^{-1}(X_i)$ form an open affine covering of Y and pre-images commute with intersection, a double application of Lemma 6.5 yields

$$[Y] = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|} [f^{-1}(X_I)] = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|} [X_I] \cdot [Z] = [X] \cdot [Z]$$

in $\mathbf{Gr}(\mathbf{subSch}_\kappa)$. \square

9.10. Theorem. *If X is a d -dimensional smooth κ -scheme and $\bar{\mathfrak{z}} \subseteq \mathfrak{z}$ a closed immersion of fat points, then the canonical map $\nabla_{\mathfrak{z}} X \rightarrow \nabla_{\bar{\mathfrak{z}}} X$ is a locally trivial fibration with fiber $\mathbb{A}_\kappa^{d(l-\bar{l})}$, where l and \bar{l} are the respective lengths of \mathfrak{z} and $\bar{\mathfrak{z}}$. In particular,*

$$[\nabla_{\mathfrak{z}} X] = [X] \cdot \mathbb{L}^{d(l-1)}$$

in $\mathbf{Gr}(\mathbf{subSch}_\kappa)$.

Proof. Let R and \bar{R} be the Artinian local coordinate rings of \mathfrak{z} and $\bar{\mathfrak{z}}$ respectively. Since arcs can be calculated locally (see the discussion following Lemma 6.4), we may assume X is the (affine) closed subscheme of \mathbb{A}_κ^m with ideal of definition $(f_1, \dots, f_s)\kappa[x]$. Since the composition of locally trivial fibrations is again a locally trivial fibration, with general fiber the product of the fibers, we may reduce to the case that $\bar{R} = R/\alpha R$ with α an element in the socle of R , that is to say, such that $\alpha \mathfrak{m} = 0$, where \mathfrak{m} is the maximal ideal of R . Let Δ be a basis of R as in Remark 9.2, with $\alpha_{l-1} = \alpha$ (since α is a socle element, such a basis always exists). In particular, $\Delta - \{\alpha\}$ is a basis of \bar{R} . We will use these bases to calculate both arc maps.

To calculate a general fiber of the map $s: \nabla_{\mathfrak{z}} X \rightarrow \nabla_{\bar{\mathfrak{z}}} X$, fix a fat point \mathfrak{w} with coordinate ring S , and a \mathfrak{w} -rational point $\bar{b}: \mathfrak{w} \rightarrow \nabla_{\bar{\mathfrak{z}}} X$, given by a tuple $\bar{\mathbf{u}}$ over S . The fiber $s(\mathfrak{w})^{-1}(\bar{b})$, is equal to the fiber of $X(\mathfrak{z}\mathfrak{w}) \rightarrow X(\bar{\mathfrak{z}}\mathfrak{w})$ above \bar{a} , where $\bar{a}: \bar{\mathfrak{z}}\mathfrak{w} \rightarrow X$ is the $\bar{\mathfrak{z}}\mathfrak{w}$ -rational point corresponding to \bar{b} , that is to say, the composition $\bar{\mathfrak{z}}\mathfrak{w} \rightarrow \bar{\mathfrak{z}} \times \nabla_{\bar{\mathfrak{z}}} X \rightarrow X$ given by Theorem 8.8 (see Corollary 8.9). Being a rational point, \bar{a} corresponds therefore to a solution \mathbf{u} in $\bar{R} \otimes_\kappa S$ of the equations $f_1 = \dots = f_s = 0$, where the relation with the tuple $\bar{\mathbf{u}}$ is given by equation (29). Let x be the center of \bar{a} , that is to say, the closed point given as the image of \bar{a} under the canonical map $X(\bar{\mathfrak{z}}\mathfrak{w}) \rightarrow X(\kappa)$. Since X is smooth at x , the Jacobian $(s \times n)$ -matrix $\text{Jac}_X := (\partial f_i / \partial x_j)$ has rank $m - d$ at x . Replacing X by an affine local neighborhood of x and rearranging the variables if necessary, we may assume that the first $(m - d) \times (m - d)$ -minor in Jac_X is invertible on X .

The surjection $R \rightarrow \bar{R}$ induces a surjection $R \otimes_\kappa S \rightarrow \bar{R} \otimes_\kappa S$. The fiber above \bar{a} is therefore defined by the equations $f_j(\mathbf{u} + \tilde{x}_{l-1}\alpha) = 0$, for $j = 1, \dots, s$. By Taylor expansion, this becomes

$$(41) \quad 0 = f_j(\mathbf{u} + \tilde{x}_{l-1}\alpha) = \left(\sum_{i=1}^m \frac{\partial f_j}{\partial x_i}(\mathbf{u}) \tilde{x}_{l-1,i} \right) \alpha$$

since $f_j(\mathbf{u}) = 0$ and $\alpha^2 = 0$ in $R \otimes_\kappa S$. In fact, since $\mathbf{u} \equiv \bar{\mathbf{u}}_0 \pmod{\mathfrak{m}(R \otimes_\kappa S)}$ and $\alpha \mathfrak{m} = 0$, we may replace each $\partial f_j / \partial x_i(\mathbf{u})$ in (41) by $\partial f_j / \partial x_i(\bar{\mathbf{u}}_0)$. Hence, the fiber above

\bar{a} is the linear subspace of $(R \otimes S)^m$ defined as the kernel of the Jacobian $\text{Jac}_X(\tilde{\mathbf{u}}_0)$. In view of the shape of the Jacobian of X , we can find $g_{ij} \in \kappa[x]$ such that

$$\tilde{x}_{l-1,i} = \sum_{j>m-d} g_{ij}(\tilde{\mathbf{u}}_0) \tilde{x}_{l-1,j}$$

for all $i \leq m-d$, by Kramer's rule. Therefore, viewing the parameter $\tilde{\mathbf{u}}_0$ as varying over $X(\mathfrak{w})$, the fiber of $s(\mathfrak{w})$ is the constant space \mathbb{A}_κ^d , as we needed to show.

Applying this to $\nabla_{\mathfrak{z}} X \rightarrow X$, (note that $X = \nabla_\kappa X$) we get a locally trivial fibration with fiber equal to $\mathbb{A}_\kappa^{d(l-1)}$, so that the last assertion follows from Lemma 9.9. \square

Calculations, like for instance Example 9.6, suggest that even for certain non-reduced schemes, there might be an underlying locally trivial fibration. Based on these examples, I would venture the following conjecture (here we write X^{red} for the underlying reduced variety of a scheme X):

9.11. *Question.* Let \mathfrak{z} be a fat point of length l and X a d -dimensional κ -scheme. If the reduction of X is smooth, when is the induced reduction map $(\nabla_{\mathfrak{z}} X)^{\text{red}} \rightarrow X^{\text{red}}$ a locally trivial fibration with fiber \mathbb{A}_κ^m , for some m .

9.12. *Remark.* As we shall see in Example 10.2 below, m can be different from $d(l-1)$, the value that we get in the reduced case. In many cases, the answer seems to be affirmative, but there are exceptions, see Example 9.13 below.

Moreover, as can be seen from Table (1) below, taking arcs does not commute with reduction, that is to say, $(\nabla_{\mathfrak{z}} X)^{\text{red}}$ is in general not equal to the arc space $\nabla_{\mathfrak{z}}(X^{\text{red}})$ of the reduction of X , nor even to the reduction of the latter arc space.

9.13. **Example.** The simplest instance to which Question 9.11 applies is when X itself is a fat point \mathfrak{x} . The expectation then is that

$$(42) \quad (\nabla_{\mathfrak{z}} \mathfrak{x})^{\text{red}} \cong \mathbb{A}_\kappa^m$$

for some m (for expected values, see Example 10.2 below). Example 9.6 provides instances in which (42) holds. However, the following is a counterexample: let $\mathfrak{z} := J_O^4 C$, where C is the cuspidal curve with equation $\xi^2 - \zeta^3 = 0$ and O the origin, its unique singularity. Let us calculate its *auto-arcs* $\nabla_{\mathfrak{z}} \mathfrak{z}$. As the monomials in ξ and ζ of degree at most two together with $\xi\zeta^2$ form a basis of the coordinate ring R of \mathfrak{z} , its length is 7 and the generic arcs are

$$\hat{x} = \tilde{x}_0 + \tilde{x}_1\xi + \cdots + \tilde{x}_6\xi\zeta^2 \quad \text{and} \quad \hat{y} = \tilde{y}_0 + \tilde{y}_1\xi + \cdots + \tilde{y}_6\xi\zeta^2.$$

Since the arc scheme $\nabla_{\mathfrak{z}} \mathfrak{z}$ lies above the origin, its reduction lies in the subvariety of \mathbb{A}_κ^{14} defined by $\tilde{x}_0 = \tilde{y}_0 = 0$, and hence, we may put these two to zero in the generic arcs and work inside the affine space \mathbb{A}_κ^{12} given by the remaining arc variables. From the fact that $\xi^3 = 0$ in R , the arc scheme is contained in the closed subscheme of \mathbb{A}_κ^{12} by the coefficients of the expansion of

$$\hat{x}^3 = 3\tilde{x}_1\tilde{x}_2^2\xi\zeta^2 + \tilde{x}_2^3\xi^2 + \dots$$

In particular, since \tilde{x}_2^2 vanishes, the reduction lies in the subvariety given by $\tilde{x}_2 = 0$, and so we may again put this variable equal to zero and work in the corresponding 11-dimensional affine space. The remaining equations come from the expansion of

$$\begin{aligned} \hat{x}^2 - \hat{y}^3 &= (\tilde{x}_1\xi + \tilde{x}_3\xi^2 + \tilde{x}_4\xi\zeta + \tilde{x}_5\xi^2 + \dots)^2 - (\tilde{y}_1\xi + \tilde{y}_2\xi + \dots)^3 \\ &= (\tilde{x}_1^2 - \tilde{y}_2^3)\xi^2 + (2\tilde{x}_1\tilde{x}_5 - 3\tilde{y}_1\tilde{y}_2^2)\xi\zeta^2 \end{aligned}$$

showing that the reduced arc space is the singular variety with equations $\tilde{x}_1^2 - \tilde{y}_2^3 = 2\tilde{x}_1\tilde{x}_5 - 3\tilde{y}_1\tilde{y}_2^2 = 0$. Note that the latter can be viewed as the tangent bundle of the cusp. More precisely, instead of the anticipated (42), we obtain the following modified form of the auto-arc variety

$$(\nabla_{J_O^4 C}(J_O^4 C))^{\text{red}} \cong \nabla_{l_2} C \times \mathbb{A}_{\kappa}^7,$$

a singular 9-dimensional variety. However, I could not find such a form for values higher than 4.

Locally constructible sieves. We say that a sieve \mathfrak{X} on a κ -scheme X is *locally constructible*, if $\mathfrak{X}(\mathfrak{z})$ is constructible in $X(\mathfrak{z})$, for each fat point \mathfrak{z} , by which we mean that $\nabla_{\mathfrak{z}} \mathfrak{X}(\kappa)$ is constructible in the Zariski topology on the variety $\nabla_{\mathfrak{z}} X(\kappa)$ viewed as the space of closed points of $\nabla_{\mathfrak{z}} X$.

9.14. Proposition. *Any formal motif is locally constructible.*

Proof. This follows from Chevalley's theorem and Theorem 9.7 in case \mathfrak{X} is sub-schematic, since, for a morphism $\varphi: Y \rightarrow X$ of κ -schemes, $\text{Im}(\varphi)(\mathfrak{z})$, as a subset of $\nabla_{\mathfrak{z}} X$, is the image of the map $\nabla_{\mathfrak{z}} Y(\kappa) \rightarrow \nabla_{\mathfrak{z}} X(\kappa)$. The formal case then follows from this, since there exists a sub-schematic motif $\mathfrak{Y} \subseteq \mathfrak{X}$ such that $\mathfrak{Y}(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z})$. \square

10. DIMENSION

In this section, we assume κ is an algebraically closed field. The dimension of an arc scheme $\nabla_{\mathfrak{z}} X$ is a subtle invariant depending on \mathfrak{z} and X , and not just on their respective length l and dimension d ; see Table (1) below. The underlying cause for this phenomenon is the fact that taking reduction does not commute with taking arcs. To exemplify this behavior, we list, for small lengths, some defining equations of arcs and their reductions for three different closed subschemes X with the same underlying one-dimensional variety, the union of two lines in the plane. Here l_i denotes the closed point with coordinate ring $\kappa[\xi]/\xi^l \kappa[\xi]$, that is to say, the l -th germ of the origin on the affine line.

TABLE 1. Dimension δ and equations of arcs and their reductions.

X	l	$xy = 0$	δ	$x^2y = 0$	δ	$x^2y^3 = 0$	δ
∇_{l_i}	1	$\tilde{x}_0\tilde{y}_0,$		$\tilde{x}_0^2\tilde{y}_0,$		$\tilde{x}_0^2\tilde{y}_0^3,$	
	2	$\tilde{x}_0\tilde{y}_1 + \tilde{x}_1\tilde{y}_0,$		$2\tilde{x}_0\tilde{x}_1\tilde{y}_0 + \tilde{x}_0^2\tilde{y}_1,$		$2\tilde{x}_0\tilde{x}_1\tilde{y}_0^3 + 3\tilde{x}_0^2\tilde{y}_0^2\tilde{y}_1,$	
	3	$\tilde{x}_0\tilde{y}_2 + \tilde{x}_1\tilde{y}_1 + \tilde{x}_2\tilde{y}_0$		$\tilde{x}_0^2\tilde{y}_2 + 2\tilde{x}_0\tilde{x}_1\tilde{y}_1 + (2\tilde{x}_0\tilde{x}_2 + \tilde{x}_1^2)\tilde{y}_0$		$3\tilde{x}_0^2(\tilde{y}_0\tilde{y}_1^2 + \tilde{y}_0^2\tilde{y}_2) + 6\tilde{x}_0\tilde{x}_1\tilde{y}_0^2\tilde{y}_1 + (\tilde{x}_1^2 + 2\tilde{x}_0\tilde{x}_2)\tilde{y}_0^3$	
$\nabla_{l_i}^{\text{red}}$	1	$\tilde{x}_0\tilde{y}_0,$	1	$\tilde{x}_0\tilde{y}_0,$	1	$\tilde{x}_0\tilde{y}_0,$	1
	2	$\tilde{x}_0\tilde{y}_1, \tilde{x}_1\tilde{y}_0,$	2	$\tilde{x}_0\tilde{y}_1,$	3	[no new equation]	3
	3	$\tilde{x}_0\tilde{y}_2, \tilde{x}_1\tilde{y}_1, \tilde{x}_2\tilde{y}_0$	3	$\tilde{x}_0\tilde{y}_2, \tilde{x}_1\tilde{y}_0$	4	$\tilde{x}_1\tilde{y}_0$	5

As substantiated by the data in this table, we have the following general estimate:

10.1. Lemma. *The dimension of $\nabla_{\mathfrak{z}} X$ is at least dl , where d is the dimension of X and l the length of \mathfrak{z} . If X is a variety, then this is an equality.*

Proof. Assume first that X is an irreducible variety, so that it contains a dense open subset U which is non-singular. By Theorem 9.3, the pull-back $\nabla_{\mathfrak{z}}U = U \times_X \nabla_{\mathfrak{z}}X$ is a dense open subset of $\nabla_{\mathfrak{z}}X$. Moreover, by Theorem 9.10 the dimension of $\nabla_{\mathfrak{z}}U$ is equal to dl , whence, by density, so is that of $\nabla_{\mathfrak{z}}X$. If X is only reduced, then we may repeat this argument on an irreducible component of X , and using once more the openness of arcs, conclude that $\nabla_{\mathfrak{z}}X$ has dimension dl .

For X arbitrary, let $V := X^{\text{red}}$ be the variety underlying X . The closed immersion $V \subseteq X$ yields a closed immersion $\nabla_{\mathfrak{z}}V \subseteq \nabla_{\mathfrak{z}}X$ by Corollary 8.9. The result now follows from the reduced case applied to V . \square

We will call the difference $\dim(\nabla_{\mathfrak{z}}X) - dl$ the *defect of X at \mathfrak{z}* . Varieties therefore have no defect. The bound given by Lemma 10.1 is far from optimal, as can be seen by taking the arc scheme of a fat point (see, for instance, Example 9.6). The growth of the dimension of auto-arcs (see Example 9.13), that is to say, the function

$$\delta(\mathfrak{z}) := \dim(\nabla_{\mathfrak{z}}\mathfrak{z})$$

for \mathfrak{z} a fat point, is still quite puzzling. By Example 9.6, we have $\delta(l_n) = n - 1$. However, the next example shows that $\delta(\mathfrak{z})$ can be bigger than $\ell(\mathfrak{z})$.

10.2. Example. Let $\mathfrak{o}_n := J_{\mathcal{O}_\kappa}^n \mathbb{A}_\kappa^2$ be the n -fold origin in the plane with ideal of definition $(\xi, \zeta)^n$. Its length is equal to $o_n := \binom{n+1}{2}$, with a basis consisting of all monomials in ξ and ζ of degree strictly less than n . Let

$$(43) \quad \hat{x} := \sum_{i+j < n} \tilde{x}_{ij} \xi^i \zeta^j \quad \text{and} \quad \hat{y} := \sum_{i+j < n} \tilde{y}_{ij} \xi^i \zeta^j$$

be the generic arcs, so that $\nabla_{\mathfrak{o}_n} \mathfrak{o}_n$ is the closed subscheme of $\mathbb{A}_\kappa^{\mathfrak{o}_n}$ given by the coefficients of the monomials $\hat{x}^i \hat{y}^{n-i}$, for $i = 0, \dots, n$. Since the arc scheme $\nabla_{\mathfrak{o}_n} \mathfrak{o}_n$ lies above the origin, its defining equations contain the ideal $(\tilde{x}_{00}, \tilde{y}_{00})^n$. To calculate its dimension, we may take its reduction, which means that we may put \tilde{x}_{00} and \tilde{y}_{00} equal to zero in (43). However, any monomial of degree n in the generic arcs is then identical zero, showing that the reduction of the arc scheme is given by $\tilde{x}_{00} = \tilde{y}_{00} = 0$, and hence, its dimension is equal to

$$\delta(\mathfrak{o}_n) = 2 \binom{n+1}{2} - 2 = n^2 + n - 2.$$

One might be tempted to propose therefore that $\delta(\mathfrak{z})$ is equal to the embedding dimension of \mathfrak{z} times its length minus one, but the next example disproves this. Namely, without proof, we state that $\delta(\mathfrak{z}) = 7$ for \mathfrak{z} the fat point in the plane with equations $\xi^2 = \xi\zeta^2 = \zeta^3 = 0$ (note that \mathfrak{z} has length 5 and embedding dimension 2, so that the expected value would be $2 \times 4 = 8$). Note that the auto-arc space $\nabla_{\mathfrak{z}}\mathfrak{z}$ is often, but not always an affine space (see Question 9.11 and the example following it).

It seems plausible that $\delta(J_P^n Y)$ grows as a polynomial in n of degree d , for any d -dimensional closed germ (Y, P) . In particular, we expect the limit

$$e(Y, P) := \lim_{n \rightarrow \infty} \frac{\delta(J_P^n Y)}{\ell(J_P^n Y)}$$

to exist. For instance, an easy extension of the above examples yields $e(\mathbb{A}_\kappa^m, O) = m$. In view of Question 9.11, we would even expect that the *auto-Igusa-zeta* series

$$\zeta_{\mathfrak{z}}(t) := \sum_{n=1}^{\infty} \mathbb{L}^{-d\ell(J_P^n Y)} [\nabla_{J_P^n Y}(J_P^n Y)] t^n$$

is rational over the localization of the classical Grothendieck ring with respect to \mathbb{L} , for any d -dimensional closed germ (Y, P) , a phenomenon that we will study in §14 below under the name of *motivic rationality* (and where we also explain the choice of power of \mathbb{L}). What about its motivic rationality over the localization $\mathbf{Gr}(\mathbf{Form}_\kappa)_{\mathbb{L}}$ of the formal Grothendieck ring?

Dimension of a motif. Given a formal motif \mathfrak{X} on a κ -scheme X , we define its *dimension* as the dimension of $\mathfrak{X}(\kappa)$. This is well-defined since $\mathfrak{X}(\kappa)$ is a constructible subset of $X(\kappa)$ by Proposition 9.14. If $\mathfrak{X} = X^\circ$ is representable, then its dimension is precisely the dimension of the scheme X . On the other hand, if \mathfrak{X} is the formal completion of X at a closed point, then \mathfrak{X} has dimension zero, whereas its global section ring has dimension equal to that of X at P by Corollary 7.6.

10.3. Proposition. *If two formal motives have the same class in $\mathbf{Gr}(\mathbf{Form}_\kappa)$, then they have the same dimension.*

Proof. Since dimension is determined by the κ -rational points, we may take, using Theorem 7.4, the image of this common class in $\mathbf{Gr}(\mathbf{Var}_\kappa)$, where the result is known to hold. \square

As we will work over $\mathbf{G} := \mathbf{Gr}(\mathbf{Form}_\kappa)_{\mathbb{L}}$ below, we extend the notion of dimension into an integer valued invariant on this localized Grothendieck ring by defining the dimension of $[\mathfrak{X}] \cdot \mathbb{L}^{-i}$ to be $\dim(\mathfrak{X}) - i$, for any formal motif \mathfrak{X} and any $i \in \mathbb{Z}$. In particular, if X has dimension d and \mathfrak{z} length l , then $[\nabla_{\mathfrak{z}} X] \cdot \mathbb{L}^{-dl}$ has positive dimension, which is the reason behind the introduction of this power of the Lefschetz class in the formulas below. This also gives us the Kontsevich filtration by dimension on \mathbf{G} . Namely, for each $m \in \mathbb{N}$, let $\Gamma_m(\mathbf{G})$ be the subgroup generated by all classes $[\mathfrak{X}] \cdot \mathbb{L}^{-i}$ of dimension at most $-m$. This is a descending filtration and the completion of \mathbf{G} with respect to this filtration will be denoted $\hat{\mathbf{G}}$. However, since we define motivic filtration locally (see §15 below), we will not make use of it.

11. DEFORMED ARCS

We continue with the setup from §9: let $j_{\mathfrak{z}} : \mathfrak{z} \rightarrow \mathrm{Spec} \kappa$ be the structure morphism of a fat point \mathfrak{z} over an algebraically closed field κ . Instead of looking at the double adjunction giving rise to the arc functor $\nabla_{\mathfrak{z}}$, we consider here the diminution part only, that is to say, the right adjoint $\nabla_{j_{\mathfrak{z}}}^*$ satisfying for each \mathfrak{z} -sieve \mathfrak{Y} on a \mathfrak{z} -scheme Y and each κ -fat point \mathfrak{w} , an isomorphism

$$(\nabla_{j_{\mathfrak{z}}}^* \mathfrak{Y})(\mathfrak{w}) \cong \mathfrak{Y}(j_{\mathfrak{z}}^* \mathfrak{w})$$

where this time, we have to view $j_{\mathfrak{z}}^* \mathfrak{w} = \mathfrak{z} \mathfrak{w}$ as a \mathfrak{z} -fat point. By Theorem 8.8, we associate in particular to any \mathfrak{z} -scheme Y , a κ -scheme $\nabla_{j_{\mathfrak{z}}}^* Y$. In particular, if $Y = j_{\mathfrak{z}}^* X$ is obtained from a κ -scheme X by base change, then

$$(44) \quad \nabla_{j_{\mathfrak{z}}}^* Y = \nabla_{\mathfrak{z}} X$$

by Corollary 8.9.

Apart from $j_{\mathfrak{z}}$, we also have the residue field morphism $\pi_{\mathfrak{z}} : \mathrm{Spec} \kappa \rightarrow \mathfrak{z}$. To a \mathfrak{z} -scheme Y , we can therefore also associate the base change $\bar{Y} := \pi_{\mathfrak{z}}^* Y$, called the *closed fiber* of Y . We can think of Y as a *fat deformation* of $j_{\mathfrak{z}}^* \bar{Y}$. Indeed, since $\kappa \times_{\mathfrak{z}} \kappa = \kappa$, any κ -rational point of Y is also a κ -rational point on \bar{Y} , that is to say, $Y(\kappa) = \bar{Y}(\kappa) = j_{\mathfrak{z}}^* \bar{Y}(\kappa)$, showing that Y and $j_{\mathfrak{z}}^* \bar{Y}$ have the same underlying variety.

11.1. Example. For instance, if C is the curve $x^2 - y^3$ and \mathfrak{l}_n the fat point with coordinate ring $R_n := \kappa[\xi]/\xi^n \kappa[\xi]$, then the \mathfrak{l}_3 -scheme X with coordinate ring $R_3[x, y]/(x^2 - y^3 - \xi^2)R_3[x, y]$ has closed fiber C , and $X(\kappa) = C(\kappa)$. Note however that $X(\mathfrak{l}_3) \neq C(\mathfrak{l}_3)$. In fact, truncation yields a map $X(\mathfrak{l}_3) \rightarrow C(\mathfrak{l}_2)$.

Hence, by (44), we may likewise think of $\nabla_{j_3^*} Y$ as a fat deformation of the arc space $\nabla_3 \bar{Y}$ of its closed fiber, justifying the term *deformed arc space* for $\nabla_{j_3^*} Y$. This construction is compatible then with specializations in the following sense. Fix a κ -scheme Z . The base change $j_Z: Z \times \mathfrak{z} \rightarrow Z$ is again a finite, flat homomorphism satisfying condition (\dagger), thus allowing us to consider the diminution $\nabla_{j_Z^*}$, associating to any $Z \times \mathfrak{z}$ -scheme Y , a Z -scheme $\nabla_{j_Z^*} Y$, called the *relative arc scheme* of Y . The deformed arc space is then given by the special case when $Z = \text{Spec } \kappa$.

11.2. Proposition. *Let \mathfrak{z} be a κ -fat point and Z a κ -scheme. For every $Z \times \mathfrak{z}$ -scheme Y , viewed as a family over Z in the sense of §8.15, and for any κ -rational point a on Z , we have an isomorphism*

$$\nabla_{j_3^*}(Y_{\tilde{a}}) = (\nabla_{j_Z^*} Y)_a$$

of κ -schemes, where $\tilde{a}: \mathfrak{z} \rightarrow Z \times \mathfrak{z}$ is the base change of a .

Proof. Immediately from Theorem 8.11 applied to the base change diagram

$$(45) \quad \begin{array}{ccc} \mathfrak{z} & \xrightarrow{j_3} & \text{Spec } \kappa \\ \tilde{a} \downarrow & & \downarrow a \\ Z \times \mathfrak{z} & \xrightarrow{j_Z} & Z. \end{array}$$

□

So, returning to Example 11.1, let $X \subseteq \mathbb{A}_{\mathfrak{l}_3}^3$ be the hypersurface with equation $x^2 - y^3 - z\xi^2$. As a family over $\mathbb{A}_{\mathfrak{l}_3}^1$ via projection on the last coordinate, its specializations X_a are all isomorphic if $a \neq 0$, whereas the special fiber $X_0 = C \times \mathfrak{l}_3$. The corresponding relative arc scheme $\nabla_{j_{\mathbb{A}_{\mathfrak{l}_3}^1, \mathfrak{l}_3}^*} X$ is given by

$$\tilde{x}_0^2 - \tilde{y}_0^3 = 2\tilde{x}_0\tilde{y}_1 - 3\tilde{y}_0^2\tilde{y}_1 = 2\tilde{x}_0\tilde{x}_2 + \tilde{x}_1^2 - 3\tilde{y}_0\tilde{y}_1^2 - 3\tilde{y}_0^2\tilde{y}_2 - \tilde{z}_0 = 0$$

Its specializations are again all isomorphic (to the third order Milnor fiber; see below) whereas the special fiber is isomorphic to $\nabla_{\mathfrak{l}_3} C$.

12. LIMIT POINTS

The closed subscheme relation defines a partial order relation on $\mathbb{F}\text{at}_\kappa$, that is to say, we say that $\bar{\mathfrak{z}} \leq \mathfrak{z}$ if and only if $\bar{\mathfrak{z}}$ is a closed subscheme of \mathfrak{z} (and not just isomorphic to one). We already discussed direct limits with respect to the induced ordering on disjoint unions of fat points in Lemma 2.8. Here we will investigate more closely the direct limit of fat points themselves. We will assume that such a direct system contains a least element. It follows that all fat points in the system must have the same center (to wit, the center of the least element). In other words, any fat point in the directed system has the same underlying closed point, and so we will call such a system a *point system*.

We want to adjoin to the category of fat points its direct limits, but the problem is that the category of schemes is not closed under direct limits either. However, the category of locally ringed spaces is: if (X_i, \mathcal{O}_{X_i}) form a direct system, then their direct limit is the topological space $X := \varinjlim X_i$ endowed with the structure sheaf $\mathcal{O}_X := \varinjlim \mathcal{O}_{X_i}$. Since we will assume that all fat points have the same underlying topological space, namely a single point, the construction simplifies: the direct limit of a point system is simply the one-point space with its unique stalk given as the inverse limit of all the coordinate rings of the fat points in the system. As already indicated in Footnote (6), a *morphism* in this setup will mean a morphism of locally ringed spaces with values in the category of κ -algebras. For example, if R is any κ -algebra and \mathfrak{o} the locally ringed space with underlying set the origin and (unique) stalk R , and if $X = \text{Spec } A$ is any affine scheme, then $\text{Mor}_\kappa(\mathfrak{o}, X)$ is in one-one correspondence with the set of κ -algebra homomorphisms $\text{Hom}_{\kappa\text{-alg}}(A, R)$.

Let $\mathbb{X} \subseteq \mathbb{F}\text{at}_\kappa$ be a point system. Its direct limit $\varinjlim \mathbb{X}$, as a one-point locally ringed space, is called a *limit point*. Some examples of limit points are:

- (1) If \mathbb{X} is finite, the direct limit is just its maximum, whence a fat point.
- (2) Given a closed germ (Y, P) , its formal completion \hat{Y}_P is the direct limit of the jets $J_P^n Y$, whence a limit point.
- (3) The direct limit of all fat points with the same center is called the *universal point* and is denoted \mathfrak{u}_κ , or just \mathfrak{u} . Any limit point admits a closed immersion into \mathfrak{u} . In particular, up to isomorphism, \mathfrak{u} does not depend on the underlying point.

12.1. Lemma. *The stalk of the universal point \mathfrak{u}_κ is isomorphic to the power series ring over κ in countably many indeterminates.*

Proof. Any fat point is a closed subscheme of some formal scheme $(\widehat{\mathbb{A}_\kappa^n})$. Hence suffices to show that the inverse limit of the power series rings $S_n := \kappa[[x_1, \dots, x_n]]$ under the canonical projections $S_m \rightarrow S_n$ given by modding out the variables x_i for $n < i \leq m$ is isomorphic to the power series ring $\kappa[[x]]$ in countably many indeterminates $x = (x_1, x_2, \dots)$. To this end, let $f_n \in S_n$ be a compatible sequence in the inverse system. For each exponent $\nu = (\nu_1, \nu_2, \dots)$ in the direct sum $\mathbb{N}^{(\mathbb{N})}$ of countably many copies of \mathbb{N} , and each n , let $a_{n,\nu} \in \kappa$ be the coefficient of $x^\nu := x_1^{\nu_1} \cdots x_{i(\nu)}^{\nu_{i(\nu)}}$ in f_n , where $i(\nu)$ is the largest index for which ν_i is non-zero. Compatibility means that there exists for each ν an element $a_\nu \in \kappa$ such that $a_\nu = a_{n,\nu}$ for all $n > i(\nu)$. Hence $f := \sum_\nu a_\nu x^\nu \in \kappa[[x]]$ is the limit of the sequence f_n , proving the claim. \square

12.2. Remark. I would guess that, similarly, the direct limit of all disjoint unions of fat points is isomorphic to the affine space \mathbb{A}_κ^ω of countable dimension.

To make the limit points into a category, denoted $\widehat{\mathbb{F}\text{at}_\kappa}$, take morphisms to be direct limits of morphisms of fat points. More precisely, given point systems $\mathbb{X}, \mathbb{Y} \subseteq \mathbb{F}\text{at}_\kappa$ with respective direct limits \mathfrak{x} and \mathfrak{y} , then a morphism (of locally ringed spaces) $\varphi: \mathfrak{x} \rightarrow \mathfrak{y}$ is a *morphism of limit points* if for each $\mathfrak{z} \in \mathbb{X}$ there exists a $\mathfrak{v} \in \mathbb{Y}$ such that $\varphi(\mathfrak{z}) \subseteq \mathfrak{v}$, or, dually, if the induced morphism $\varinjlim \mathcal{O}_{\mathfrak{v}} \rightarrow \varinjlim \mathcal{O}_{\mathfrak{z}}$ has the property that for each $\mathfrak{z} \in \mathbb{X}$, we can find a $\mathfrak{v} \in \mathbb{Y}$ such that this morphism factors through $\mathcal{O}_{\mathfrak{v}} \rightarrow \mathcal{O}_{\mathfrak{z}}$. In this way, the category $\widehat{\mathbb{F}\text{at}_\kappa}$ of limit points is an extension of the category $\mathbb{F}\text{at}_\kappa$ of fat points, which in a sense acts as its *compactification*. In particular, any limit point \mathfrak{x} admits a canonical structure morphism $j_\mathfrak{x}: \mathfrak{x} \rightarrow \text{Spec } \kappa$. We also extend the partial order relation on $\mathbb{F}\text{at}_\kappa$ to one on $\widehat{\mathbb{F}\text{at}_\kappa}$ as follows. Firstly, we say that $\mathfrak{z} \leq \mathfrak{x}$ for \mathfrak{z} a fat point and $\mathfrak{x} = \varinjlim \mathbb{X}$ a limit point, if $\mathfrak{z} \leq \mathfrak{v}$ for some fat point $\mathfrak{v} \in \mathbb{X}$. It follows that there is a canonical embedding $\mathfrak{z} \subseteq \mathfrak{x}$ which induces a surjection on the stalks, and which we therefore call a *closed embedding*

in analogy with the scheme-theoretic concept. We then say for a limit point $\mathfrak{y} = \varinjlim \mathbb{Y}$ that $\mathfrak{y} \leq \mathfrak{x}$ if for every $\mathfrak{z} \in \mathbb{Y}$ we have $\mathfrak{z} \leq \mathfrak{x}$. It follows that we have a canonical morphism of limit points $\mathfrak{y} \rightarrow \mathfrak{x}$ which is again surjective on their stalks, and hence can rightly be called once more a *closed embedding*. One checks that this defines indeed a partial order on limit points extending the one on fat points.

We call a limit point \mathfrak{x} *bounded* if it is the direct limit of fat points of embedding dimension at most n , for some n . Formal completions of closed germs are examples of bounded limit points, whereas \mathfrak{u} clearly is not. In fact, any bounded limit point arises in a similar, analytical way:

12.3. Proposition. *The bounded limit points are in one-one correspondence with analytic germs. More precisely, the stalks of bounded limit points are precisely the complete Noetherian local rings with residue field κ .*

Proof. Let \mathfrak{x} be a bounded limit point, say, realized as the direct limit of fat points $\mathfrak{z} \subseteq \mathbb{A}_\kappa^n$ centered at the origin, for some fixed n . Let $\kappa[x]/\mathfrak{a}_\mathfrak{z}$ be the coordinate ring of \mathfrak{z} , so that $\mathfrak{a}_\mathfrak{z} \subseteq \kappa[x]$ is \mathfrak{m} -primary, where \mathfrak{m} is the maximal ideal generated by the variables. Let I be the intersection of all $\mathfrak{a}_\mathfrak{z} \subseteq \kappa[x]$. I claim that \mathfrak{x} has stalk equal to $S := \kappa[[x]]/I$. Indeed, by a theorem of Chevalley ([12, Exercise 8.7]), there exists for each i some ideal of definition \mathfrak{a} of one of the fat points in the direct system such that $\mathfrak{a}S \subseteq \mathfrak{m}^i S$. In particular, the inverse limit is simply the \mathfrak{m} -adic completion of S , which is of course S itself.

The converse is also obvious: given a complete Noetherian ring (S, \mathfrak{m}) with residue field κ , then by Cohen's structure theorem, it is of the form $\kappa[[x]]/I$ for some ideal I . One easily checks that it is the coordinate ring of the direct limit of the corresponding jets $\text{Spec}(S/\mathfrak{m}^n)$. \square

Any limit point $\mathfrak{x} = \varinjlim \mathbb{X}$ defines a presheaf \mathfrak{x}° by assigning to a fat point \mathfrak{z} the set of morphisms $\text{Mor}_{\widehat{\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa}}(\mathfrak{z}, \mathfrak{x})$.

12.4. Corollary. *The presheaf \mathfrak{x}° defined by a limit point $\mathfrak{x} = \varinjlim \mathbb{X}$ is the inverse limit of the representable functors \mathfrak{v}° for $\mathfrak{v} \in \mathbb{X}$. If \mathfrak{x} is moreover bounded, then \mathfrak{x}° is a formal motif.*

Proof. Given a fat point $\mathfrak{z} = \text{Spec } R$, we have to show that $\mathfrak{v}(\mathfrak{z})$ for $\mathfrak{v} \in \mathbb{X}$ forms an inverse system with inverse limit equal to $\text{Mor}_{\widehat{\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa}}(\mathfrak{z}, \mathfrak{x})$. Let (S, \mathfrak{m}) be the stalk of \mathfrak{x} , that is to say, the inverse limit of the coordinate rings of the fat points belonging to \mathbb{X} . The first statement is immediate by functoriality, and for the second, note that since R has finite length,

$$(46) \quad \text{Mor}_{\widehat{\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa}}(\mathfrak{z}, \mathfrak{x}) \cong \text{Hom}_{\kappa\text{-alg}}(S, R).$$

More precisely, any κ -algebra homomorphism $a: S \rightarrow R$ factors through $S/\mathfrak{m}^l \rightarrow R$, for $l = \ell(R)$. Moreover, if n is the embedding dimension of R , then there exists a complete, Noetherian residue ring \tilde{S} of S of embedding dimension at most n such that a factor as $S \rightarrow \tilde{S}/\mathfrak{m}^l \tilde{S} \rightarrow R$. By the same argument as in the proof of Proposition 12.3, there is some $\mathfrak{v} = \text{Spec } T \in \mathbb{X}$ such that $T \rightarrow \tilde{S}/\mathfrak{m}^l \tilde{S}$, showing that a is already induced by the morphism $\mathfrak{z} \rightarrow \mathfrak{v}$. In fact, if \mathfrak{x} is bounded, then we may choose \mathfrak{v} independent from a , showing that $\mathfrak{x}^\circ(\mathfrak{z}) = \mathfrak{v}(\mathfrak{z})$. Since all $\mathfrak{v} \in \mathbb{X}$ embed in the same affine space, \mathfrak{x}° is a locally schemic sieve on this space, whence a formal motif. \square

12.5. Remark. The identification (46) shows that \mathfrak{x}° is pro-representable in the sense of Footnote (6).

Let $\mathfrak{x} = \varinjlim \mathbb{X}$ be a limit point. Given a presheaf \mathfrak{X} , the collection of all $\mathfrak{X}(\mathfrak{v})$ for $\mathfrak{v} \in \mathbb{X}$ is an inverse system of sets given by the maps $i_{\mathfrak{v}, \mathfrak{w}}: \mathfrak{X}(\mathfrak{w}) \rightarrow \mathfrak{X}(\mathfrak{v})$ if $\mathfrak{v} \leq \mathfrak{w}$ in \mathbb{X} , where $i_{\mathfrak{v}, \mathfrak{w}}$ is induced by the embedding $\mathfrak{v} \subseteq \mathfrak{w}$. We denote the inverse limit of this system simply by $\mathfrak{X}(\mathfrak{x})$. It follows from the definition of morphisms of limit points that \mathfrak{X} becomes a functor on the category of limit points. In other words, any presheaf on $\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa$ extends to a presheaf on $\widehat{\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa}$; this principle will simply be called *continuity*. Since inverse limits commute with presheafs, one easily verifies that if $s: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of presheafs (=natural transformation), then for any limit point \mathfrak{x} , this induces a map $\mathfrak{X}(\mathfrak{x}) \rightarrow \mathfrak{Y}(\mathfrak{x})$, showing that extension by continuity is functorial. The extension of a (pro-)representable functor on $\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa$ to $\widehat{\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa}$ will again be called *(pro-)representable*.

If \mathbb{X} is a point system in $\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa$ with limit \mathfrak{x} , and if \mathfrak{z} is any fat point with canonical morphism $j_{\mathfrak{z}}: \mathfrak{z} \rightarrow \text{Spec } \kappa$, then the base change $j_{\mathfrak{z}}^* \mathbb{X}$ consisting of all $\mathfrak{z}\mathfrak{v}$ for $\mathfrak{v} \in \mathbb{X}$ is again a point system, whose limit we simply denote by $\mathfrak{z}\mathfrak{x}$ (the reader can check that this defines a product in the category $\widehat{\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa}$). Repeating this argument on the first factor then shows that we may even multiply any two limit points. However, this multiplication does no longer behave as well as before. For instance, since the base change $j_{\mathfrak{z}}^*(\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa)$ by any fat point \mathfrak{z} is equal to the whole category $\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa$, we get $\mathfrak{z}u = u$.

Let $j_{\mathfrak{x}}: \mathfrak{x} \rightarrow \text{Spec } \kappa$ be the structure morphism of the limit point \mathfrak{x} . Strictly speaking, as this is only a direct limit of finite, flat morphisms, the theory of diminution does not apply, and neither that of augmentation. Nonetheless, without going into details, one could develop the theory under this weaker condition, although we will only give an ad hoc argument in the case we need it. So, given a sieve \mathfrak{X} on $\widehat{\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa}$, we define $\nabla_{\mathfrak{x}} \mathfrak{X} := \nabla_{j_{\mathfrak{x}}^*} \nabla_{(j_{\mathfrak{x}})^*} \mathfrak{X}$ at a limit point \mathfrak{y} as the set $\mathfrak{X}(\mathfrak{y}\mathfrak{x})$, where we view $\mathfrak{y}\mathfrak{x}$ again as a limit point (over κ).

12.6. Lemma. *For any limit point \mathfrak{x} and any κ -scheme X , the base change $\mathfrak{x}^* X^\circ$ is pro-representable, by the so-called arc scheme $\nabla_{\mathfrak{x}} X$ along \mathfrak{x} .*

Proof. Let \mathfrak{x} be the direct limit of the directed subset $\mathbb{X} \subseteq \mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa$. Suppose first that X is affine. Since the $\nabla_{\mathfrak{w}} X$ for $\mathfrak{w} \in \mathbb{X}$ form an inverse system of affine schemes, their inverse limit is a well-defined affine scheme \tilde{X} with coordinate ring the direct limit of the coordinate rings of the arc schemes along fat points in \mathbb{X} . By continuity, it suffices to verify that $\nabla_{\mathfrak{x}} X^\circ = \tilde{X}^\circ$ on $\mathbb{F}\mathfrak{a}\mathfrak{t}_\kappa$. To this end, fix a fat point \mathfrak{z} . From

$$\begin{aligned} (\nabla_{\mathfrak{x}} X^\circ)(\mathfrak{z}) &= X(\mathfrak{x}\mathfrak{z}) = \varprojlim_{\mathfrak{w} \in \mathbb{X}} X(\mathfrak{w}\mathfrak{z}) \\ &= \varprojlim_{\mathfrak{w} \in \mathbb{X}} \text{Mor}_\kappa(\mathfrak{z}, \nabla_{\mathfrak{w}} X) \\ &= \text{Mor}_\kappa(\mathfrak{z}, \varprojlim_{\mathfrak{w} \in \mathbb{X}} \nabla_{\mathfrak{w}} X) \\ &= \text{Mor}_\kappa(\mathfrak{z}, \tilde{X}) = \tilde{X}(\mathfrak{z}), \end{aligned}$$

where we used the universal property of inverse limits in the third line, the claim now follows. The general case follows from this by the open nature of arc schemes (Theorem 9.3) and the fact that if X admits an open affine covering of cardinality N , then so does any arc scheme $\nabla_{\mathfrak{z}} X$ by base change. \square

13. EXTENDABLE ARCS

Let \hat{Y} be a formal completion, viewed as the limit point of the germs $J_O^n Y$, and let X be a κ -scheme. By Lemma 12.6, we have an associated arc scheme $\nabla_{\hat{Y}} X$. For each n , we have a canonical map $\nabla_{\hat{Y}} X \rightarrow \nabla_{J_O^n Y} X$, which in general is not surjective (it is

so, by Theorem 9.10, when X is smooth). To study this image, we make the following definitions.

Given a closed embedding $\mathfrak{v} \subseteq \mathfrak{w}$, the image sieve given by the canonical map $\nabla_{\mathfrak{w}} X \rightarrow \nabla_{\mathfrak{v}} X$ is called the *sieve of \mathfrak{w} -extendable arcs on X along \mathfrak{v}* , and will be denoted $\nabla_{\mathfrak{w}/\mathfrak{v}} X$. By construction, it is sub-schematic. Let $\nabla_n X := \nabla_{J_{\mathcal{O}}^n Y} X$, and $\nabla_{m/n} X := \nabla_{J_{\mathcal{O}}^m Y / J_{\mathcal{O}}^n Y} X$, for $m \geq n$. Since the map $\nabla_{\hat{Y}} X \rightarrow \nabla_n X$ is not of finite type, the corresponding image sieve, denoted $\nabla_{\hat{Y}/n} X$ and called the *n -th order \hat{Y} -extendable arcs on X* , may fail to be sub-schematic. We do have:

13.1. Theorem. *For each κ -scheme X , for each formal completion \hat{Y} of a closed subscheme $Y \subseteq \mathbb{A}_{\kappa}^n$ at a point, and for each n , the n -th order extendable arcs on X along this formal completion, $\nabla_{\hat{Y}/n} X$, is a formal motif.*

Proof. Without loss of generality, we may assume that \hat{Y} is the completion of Y at the origin. It is clear that $\nabla_{\hat{Y}/n} X$ is the intersection of all $\nabla_{m/n} X$, for $m \geq n$. To show that it is a formal motif, it suffices to show that its complement is locally sub-schematic. Since each $\nabla_{m/n} X$ is sub-schematic, this will follow if we can show that for each fat point \mathfrak{z} , there is some $m_{\mathfrak{z}}$ such that $\nabla_{\hat{Y}/n} X(\mathfrak{z}) = \nabla_{m_{\mathfrak{z}}/n} X(\mathfrak{z})$.

Recall that for (R, \mathfrak{m}) a quotient of a power series ring $\kappa[[\xi]]$ modulo an ideal generated by polynomials, we have uniform strong Artin Approximation, in the sense that for any polynomial system of equations $f_1 = \dots = f_s = 0$ and every n , there exists some N , such any solution of $f_1 = \dots = f_s = 0$ in R/\mathfrak{m}^N is congruent modulo \mathfrak{m}^n to a solution in R : see for instance [15, Theorem 7.1.10], where the proof is only given for the power series ring itself, but immediately generalizes to any quotient by a polynomial ideal, whence in particular to the stalk of the formal completion \hat{Y} . This means that $\nabla_{\hat{Y}/n} X(\kappa) = \nabla_{m/n} X(\kappa)$, for some $m \geq n$. To obtain a similar identity over an arbitrary fat point \mathfrak{z} , we apply the same result but replacing X by the arc scheme $\nabla_{\mathfrak{z}} X$, yielding the existence of a $m_{\mathfrak{z}} \geq n$ such that

$$\nabla_{\hat{Y}/n} X(\mathfrak{z}) = \nabla_{\hat{Y}/n} (\nabla_{\mathfrak{z}} X)(\kappa) = \nabla_{m_{\mathfrak{z}}/n} (\nabla_{\mathfrak{z}} X)(\kappa) = \nabla_{m_{\mathfrak{z}}/n} X(\mathfrak{z}),$$

as required. \square

13.2. Remark. Since we may no longer have the required strong Artin Approximation estimate, I do not know whether this result generalizes to arbitrary limit points, that is to say, is $\nabla_{\mathfrak{y}/\mathfrak{x}} X$ a formal motif, for limit points $\mathfrak{x} \leq \mathfrak{y}$. The first case to look at is when \mathfrak{x} is a fat point and \mathfrak{y} is bounded (but not a formal completion).

14. MOTIVIC GENERATING SERIES

Although we can work in greater generality, we will assume once more that our base scheme is an algebraically closed field κ .

Motivic Igusa-zeta series. For any κ -scheme X and any closed germ (Y, P) , we define the *motivic Igusa-zeta series of X along the germ (Y, P)* as the formal power series

$$\mathrm{Igu}_{X^{\mathrm{mot}}}^{(Y, P)}(t) := \sum_{n=1}^{\infty} \mathbb{L}^{-d \cdot j_P^n(Y)} [\nabla_{J_P^n Y} X] t^n$$

in $\mathrm{Gr}(\mathrm{Form}_{\kappa})_{\mathbb{L}}[[t]]$, where d is the dimension of X and $j_P^n(Y)$ the length of the n -th jet $J_P^n Y$ (which is also equal to the Hilbert-Samuel function of $\mathcal{O}_{Y, P}$ for large n). This definition generalizes the one in [1] or [3, §4], when we take the germ of a point on a line.

14.1. Theorem. *If X is a smooth κ -variety and (Y, P) a closed germ, then*

$$\mathrm{Igu}_{X_{\mathrm{mot}}}^{(Y,P)} = \frac{\mathbb{L}^{-d}[X]t}{1-t},$$

over $\mathbf{Gr}(\mathrm{Form}_\kappa)_\mathbb{L}$.

Proof. Since X is smooth, we have

$$(47) \quad [\nabla_{J_P^n Y} X] = [X] \mathbb{L}^{d(j_P^n(Y)-1)}$$

by Theorem 9.10, from which the assertion follows easily. \square

With aid of (47) applied to affine space, we can write down a more suggestive formula for the motivic Igusa zeta-series

$$\mathrm{Igu}_{X_{\mathrm{mot}}}^{(Y,P)}(t) := \sum_{n=1}^{\infty} \frac{[\nabla_{J_P^n Y} X]}{\nabla_{J_P^n Y} \mathbb{L}^d} t^n$$

where d is the dimension of X .

14.2. Conjecture. *The motivic Igusa-zeta series $\mathrm{Igu}_{X_{\mathrm{mot}}}^{(Y,P)}$ of a κ -scheme X along an arbitrary closed germ (Y, P) is rational over $\mathbf{Gr}(\mathrm{Form}_\kappa)_\mathbb{L}$.*

More generally, given any formal motif \mathfrak{X} on a κ -scheme X , we define its Igusa-zeta series along the germ (Y, P) as the formal power series

$$\mathrm{Igu}_{\mathfrak{X}_{\mathrm{mot}}}^{(Y,P)}(t) := \sum_{n=1}^{\infty} \mathbb{L}^{-d \cdot j_P^n(Y)} \cdot [\nabla_{J_P^n Y} \mathfrak{X}] t^n$$

and conjecture its rationality.

14.3. Example. The present point of view even gives interesting new results over the classical Grothendieck ring, since we may take the image of the motivic Igusa zeta series in $\mathbf{Gr}(\mathbb{A}_\kappa)$. Continuing with the calculations made in Example 9.6, let m be a positive integer and consider the image of $\mathrm{Igu}(\mathfrak{l}_m) := \mathrm{Igu}_{\mathfrak{l}_m}^{(\mathbb{A}_\kappa^1, O)}$ over the classical Grothendieck ring. This amounts to taking the reduced scheme underlying each arc scheme $\nabla_{\mathfrak{l}_n} \mathfrak{l}_m$, and as shown above, this reduction is just $\mathbb{A}_\kappa^{n - \lceil \frac{n}{m} \rceil}$. Write $n = sm - r$ for some unique $s \geq 1$ and $0 \leq r < m$, so that $\lceil \frac{n}{m} \rceil = s$. Over $\mathbf{Gr}(\mathbb{A}_\kappa)$, we have

$$\mathrm{Igu}(\mathfrak{l}_m) = \sum_{r=0}^{m-1} \sum_{s=1}^{\infty} \mathbb{L}^{sm-r-s} t^{sm-r} = \frac{\sum_{r=0}^{m-1} (\mathbb{L}t)^{-r}}{(1 - \mathbb{L}^{m-1} t^m)}.$$

In particular, whenever Question 9.11 holds affirmatively, the image of the motivic Igusa zeta series of the fat point would be rational over the classical Grothendieck ring. Skipping the easy calculations, we have for instance that

$$\mathrm{Igu}(\mathfrak{z}) = \frac{t + \mathbb{L}t^2}{(1 - \mathbb{L}t^2)}$$

where \mathfrak{z} is the fat point with coordinate ring $\kappa[x, y]/(x^2, y^2)$. Interestingly, this is also the Igusa zeta-series of the fat point with coordinate ring $\kappa[x, y]/(x^2, xy, y^2)$.

Motivic Hilbert series. Given a motivic site \mathbb{M} , we let \mathbb{M}_0 be its restriction to the subcategory of zero-dimensional schemes, that is to say, the union of all $\mathbb{M}|_Z$, where Z runs over all zero-dimensional κ -schemes. As the product of two zero-dimensional schemes is again zero-dimensional, \mathbb{M}_0 is a partial motivic site, and hence has an associated Grothendieck ring $\mathbf{Gr}_0(\mathbb{M}) := \mathbf{Gr}(\mathbb{M}_0)$, called the *Grothendieck ring of \mathbb{M} in dimension zero*. There is a natural homomorphism $\mathbf{Gr}_0(\mathbb{M}) \rightarrow \mathbf{Gr}(\mathbb{M})$, which in general will fail to be injective, as there are a priori more relations in the latter Grothendieck ring. In particular, applied to (sub-)schematic or formal motives, we get the corresponding Grothendieck rings in dimension zero $\mathbf{Gr}_0(\mathcal{Sch}_\kappa)$, $\mathbf{Gr}_0(\mathcal{subSch}_\kappa)$, and $\mathbf{Gr}_0(\mathcal{Form}_\kappa)$.

14.4. Proposition. *The schematic Grothendieck ring in dimension zero, $\mathbf{Gr}_0(\mathcal{Sch}_\kappa)$, is freely generated, as an additive group, by the isomorphism classes of fat points.*

Proof. A zero-dimensional scheme Z is a disjoint union $\mathfrak{z}_1 \sqcup \cdots \sqcup \mathfrak{z}_s$ of fat points (in a unique way). Moreover, since any fat point is strongly connected, this unique decomposition in fat points is its schematic decomposition. Hence, by (the proof of) Theorem 5.7, the image of $[Z]$ under the composition $\mathbf{Gr}_0(\mathcal{Sch}_\kappa) \rightarrow \mathbf{Gr}(\mathcal{Sch}_\kappa) \xrightarrow{\delta} \Gamma$ is $\langle \mathfrak{z}_1 \rangle + \cdots + \langle \mathfrak{z}_s \rangle$, where as before, Γ is the free Abelian group on isomorphism classes of strongly indecomposable κ -schemes. Since $[Z] = [\mathfrak{z}_1] + \cdots + [\mathfrak{z}_s]$ in $\mathbf{Gr}_0(\mathcal{Sch}_\kappa)$, this composition is an isomorphism. \square

Let (X, P) be a closed germ over κ . For t a single variable, we define the *motivic Hilbert series* as the series

$$\mathrm{Hilb}_{(X,P)}^{\mathrm{mot}} := \sum_{n=1}^{\infty} [J_P^n X] t^n$$

in $\mathbf{Gr}_0(\mathcal{Sch}_\kappa)[[t]]$. By Proposition 14.4, we may extend the length function ℓ to a homomorphism on the Grothendieck ring in dimension zero. If we extend this further to the power series ring $\mathbf{Gr}_0(\mathcal{Sch}_\kappa)[[t]]$ by letting it act on the coefficients of a power series, then $\ell(\mathrm{Hilb}_P(X))$ is a rational function in $\mathbb{Z}[[t]]$ by Hilbert-Samuel theory (it is the first difference of the classical Hilbert series of X at P). This begs the question whether $\mathrm{Hilb}_{(X,P)}^{\mathrm{mot}}$ itself is rational over $\mathbf{Gr}_0(\mathcal{Form}_\kappa)$ or $\mathbf{Gr}(\mathcal{Form}_\kappa)$, or possibly their localizations at \mathbb{L} (obviously, it will not be rational over the schematic Grothendieck rings by Proposition 14.4). Although no longer specializing to a classical series, we may also consider the more general series

$$\mathrm{Hilb}_{(X,x)}^{\mathrm{mot}} := \sum_{n=0}^{\infty} [J_x^n X] t^n$$

where x is any point on X (not necessarily closed).

14.5. Theorem. *Let κ be an algebraically closed field of cardinality 2^γ for some infinite cardinal γ (under the Generalized Continuum hypothesis this means any uncountable algebraically closed field). The assignment $\hat{X}_P \mapsto \mathrm{Hilb}_{(X,P)}^{\mathrm{mot}}$ is a complete invariant in the sense that for closed germs (X, P) and (Y, Q) over κ , their completions \hat{X}_P and \hat{Y}_Q are abstractly isomorphic (that is to say, over some subfield of κ) if and only if they have the same motivic Hilbert series in $\mathbf{Gr}_0(\mathcal{Sch}_\kappa)$.*

Proof. Immediate from Proposition 14.4 and [16, Theorem 1.1 and §8.9]. \square

Motivic Hilbert-Kunz series. Assume that κ has characteristic p . Recall that for a given closed subscheme $Y \subseteq X$, we defined in §8 its Frobenius transform in X as the pull-back $\mathbf{F}_X^* Y := \mathbf{F}_X^* X \times_X Y$ of Y along \mathbf{F}_X . We may also take the pull-back with respect to the powers \mathbf{F}_X^n of the Frobenius, yielding the n -th Frobenius transform $\mathbf{F}_X^{n*} Y$. If Y has dimension zero, then so does any of its Frobenius transforms, and so the following series, called the *motivic Hilbert-Kunz series*,

$$\mathrm{HK}_Y^{\mathrm{mot}}(X) := \sum_{n=1}^{\infty} [\mathbf{F}_X^{n*} Y] t^n$$

is a well-defined series in $\mathbf{Gr}_0(\mathcal{S}\mathrm{ch}_{\kappa})[[t]]$. Taking the length function ℓ yields the classical Hilbert-Kunz series, of which not too much is known (one expects it to be rational). Of course, we could also take Y to be of higher dimension, and get the corresponding motivic Hilbert-Kunz series in $\mathbf{Gr}(\mathcal{S}\mathrm{ch}_{\kappa})[[t]]$.

Instead of transforms we could take iterated Frobenius motives $\mathfrak{F}_Y^n := \nabla_{\mathbf{F}^n} Y$ given as the image sieve of the n -th relative Frobenius $\mathbf{F}_{\mathbb{A}_{\kappa}^s}^n \times 1_Y$, for some closed immersion $Y \subseteq \mathbb{A}_{\kappa}^s$, in case Y is affine, and by glueing for the general case, yielding a series

$$\mathrm{Fr}^{\mathrm{mot}}(Y) := \sum_{n=1}^{\infty} [\nabla_{\mathbf{F}^n} Y] t^n$$

in $\mathbf{Gr}_0(\mathcal{S}\mathrm{ch}_{\kappa})[[t]]$. Note that $\mathfrak{F}_Y^n(\kappa) = Y(\mathbf{F}^n \kappa) = Y(\kappa)$ by Theorem 8.14, so that this series becomes the rational function $[Y]/(1-t)$ in $\mathbf{Gr}(\mathcal{V}\mathrm{ar}_{\kappa})$.

Motivic Milnor series. Let (Y, P) be a closed germ with formal completion \hat{Y}_P . Proposition 7.1 exhibits \hat{Y}_P° as a formal motif by means of its jets. However, this is not the only way to locally approximate it with schemic subsieves. Given a system of parameters ξ_1, \dots, ξ_e in $\mathcal{O}_{Y,P}$ (that is to say, a tuple of length $e = \dim(\mathcal{O}_{Y,P})$ generating an ideal primary to the maximal ideal), let η_n be the fat point with coordinate ring $B_n := \mathcal{O}_{Y,P}/(\xi_1^n, \dots, \xi_e^n)\mathcal{O}_{Y,P}$, and $j_{\eta_n} : \eta_n \rightarrow \mathrm{Spec} \kappa$ the canonical morphism. The reader can check that given a fat point \mathfrak{z} , there exists some n such that $\eta_n(\mathfrak{z}) = \hat{Y}_P(\mathfrak{z})$, that is to say, \hat{Y}_P is the limit point corresponding to the direct system $\{\eta_n\}_n$ (see §12). Recall that by the Monomial Theorem, the element $(\xi_1 \cdots \xi_e)^{n-1}$ is a non-zero element in the socle of B_n (meaning that the ideal it generates has length one).

Let $X \subseteq \mathbb{A}_{\kappa}^{d+1}$ be the $(d$ -dimensional) hypersurface with equation $f(x) = 0$ and let $Y_n \subseteq \mathbb{A}_{\eta_n}^{d+1}$ be the deformed hypersurface with equation $f(x) - (\xi_1 \cdots \xi_e)^{n-1} = 0$. In other words, it is the general fiber in the family $W_n \subseteq \mathbb{A}_{\eta_n}^{d+2}$ over the last coordinate z , given by the equation $f(x) - z(\xi_1 \cdots \xi_e)^{n-1} = 0$, whereas the special fiber is just the base change $j_{\eta_n}^* X$. We define the n -th order Milnor fiber of X along the germ (Y, P) as the deformed arc space

$$M_n(X) := \nabla_{j_{\eta_n}^*} Y_n.$$

Hence, by Proposition 11.2, with $j_{\mathbb{A}_{\kappa}^{d+2}} : \mathbb{A}_{\eta_n}^{d+2} \rightarrow \mathbb{A}_{\kappa}^{d+2}$ the base change of j_{η_n} , the specializations of the relative arc scheme are

$$(48) \quad \begin{aligned} (\nabla_{j_{\mathbb{A}_{\kappa}^{d+2}}^*} W_n)_a &= \nabla_{\eta_n} X && \text{if } a \text{ is the zero section;} \\ &= M_n(X) && \text{otherwise.} \end{aligned}$$

We define the associated Milnor series

$$\mathrm{Mil}_{X^{\mathrm{mot}}}^{(Y,P)}(t) := \sum_{n=1}^{\infty} \mathbb{L}^{-d\ell(\eta_n)} [M_n(X)] t^n$$

as a power series over $\mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}$. When (Y, P) is the germ of a point on a line, we get the schematic variant of the series introduced by Denef-Loeser et al., and by (48), this series can be viewed as a deformation of the motivic Igusa-zeta series. Therefore, in view of Conjecture 14.2, we expect the motivic Milnor series also to be rational, and in fact, as a rational function, to have degree zero. Assuming this to be true, we can calculate the limit of this series when $t \rightarrow \infty$, and this conjectural limit, presumably in $\mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}$, will be called the *motivic Milnor fiber* of X along the closed germ (Y, P) .

Motivic Hasse-Weil series. Another important generating series in algebraic geometry whose rationality—proven by Dwork in [4]—is postulated to be motivic, is the Hasse-Weil series of a scheme over a finite field \mathbb{F}_q : its general coefficient is the number of rational points over the finite extensions \mathbb{F}_{q^n} . To turn this into an abstract counting principle, we use the inversion formula relating the number of degree n effective zero cycles on X to the number of rational points in an extension of degree n , and observe that the former cycles are in one-one correspondence with the rational points on the n -fold symmetric product $X^{(n)}$ of X (given as the quotient of X^n modulo the action of the symmetric group on n -tuples). Therefore, following Kapranov [8], we propose the following motivic variant, the *Motivic Hasse-Weil series*:

$$\mathrm{HW}_X^{\mathrm{mot}} := \sum_{n=0}^{\infty} [X^{(n)}] t^n,$$

as a power series over $\mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}$. Kapranov himself proved rationality of the image of this series over $\mathbf{Gr}(\mathbf{Var}_\kappa)_\mathbb{L}$, as well as a functional equation, for certain smooth, projective irreducible curves, but the general case is still open. We know from work of Larsen and Lunts on smooth surfaces ([10]), that, in general, this cannot hold over the Grothendieck ring itself: in [11], they show that rationality over the Grothendieck ring is equivalent with the complex surface having negative Kodaira dimension. It is therefore natural to conjecture the same properties for our motivic variant $\mathrm{HW}_X^{\mathrm{mot}}$.

Motivic Poincaré series. Given a closed germ (Y, P) with formal completion \hat{Y} , viewed as a limit point, and a κ -scheme X , by Theorem 13.1, we can now define the *motivic Poincaré series of X along (Y, P)* as the formal series

$$\mathrm{Poin}_{X^{\mathrm{mot}}}^{(Y, P)}(t) := \sum_{n=1}^{\infty} \mathbb{L}^{-d\ell(J_P^n Y)} [\nabla_{\hat{Y}/n} X] t^n$$

over $\mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}$, where d is the dimension of X . Denef and Loeser proved in [2] that along the germ of a point on the line, the image of this series in the localized classical Grothendieck ring is rational, provided κ has characteristic zero. It is therefore natural to postulate:

14.6. Conjecture. *For any closed germ (Y, P) and any κ -scheme X , the associated motivic Poincaré series $\mathrm{Poin}_{X^{\mathrm{mot}}}^{(Y, P)}$ is rational over $\mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}$.*

Given X and a formal completion \hat{Y} , we may ask for each n , which are the fat points \mathfrak{z} containing the n -th jet $j_n := J_O^n Y$ such that $\nabla_{\hat{Y}/j_n} X \subseteq \nabla_{\mathfrak{z}/j_n} X$, that is to say, when are \hat{Y} -extendable arcs also \mathfrak{z} -extendable? For instance, if \hat{Y} is the completion of the affine line, then by Theorem 9.10, we can extend along any jet of a non-singular germ (W, O) , since there exist closed immersions $j_n \subseteq J_O^n W \subseteq j_n^d$, where d is the dimension of (W, O) . However, I do not know whether we can extend along the fat point given by, say, $x^4 =$

$y^4 = x^3 - y^2 = 0$. For which schemes X can every \hat{Y} -extendable arc be extended along any fat point? This is true if X is smooth, but are there any other cases?

15. MOTIVIC INTEGRATION

Unlike the Kontsevitch-Denef-Loeser motivic integration, we will only define integration on the (truncated) arc schemes. We will work over the localized Grothendieck ring $\mathbf{G} := \mathbf{Gr}(\mathbb{F}\text{orm}_\kappa)_\mathbb{L}$, for κ an algebraically closed field. Before we develop the theory, we discuss a naive approach.

Motivic measure. We fix a fat point \mathfrak{z} . Our goal is to define a motivic measure $\mu_{\mathfrak{z}}$ on formal motives. To this end, we define

$$\mu_{\mathfrak{z}}(\mathfrak{X}) := [\nabla_{\mathfrak{z}}\mathfrak{X}]$$

in \mathbf{G} . In particular, this measure does not depend on the ambient space of \mathfrak{X} , only on its germ. Using Theorem 9.7, we can extend the motivic measure to an endomorphism on \mathbf{G} . We would like to normalize this measure, with the ultimate goal—which, however, we do not discuss in this paper—to make the comparison between different fat points and take limits. One way to normalize is to make the value weightless (in the sense of dimension), by

$$\bar{\mu}_{\mathfrak{z}}(\mathfrak{X}) := \frac{[\nabla_{\mathfrak{z}}\mathfrak{X}]}{\mathbb{L}^{\dim(\nabla_{\mathfrak{z}}\mathfrak{X})}}$$

This, of course, is no longer additive. Below, however, we will normalize differently, by fixing an ambient space.

Following integration theory practice, we would like to say that

$$\int 1_{\mathfrak{X}} d_{\mathfrak{z}}x := \mu_{\mathfrak{z}}(\mathfrak{X}) := [\nabla_{\mathfrak{z}}\mathfrak{X}]$$

and extend this to arbitrary step functions. Here, a *step function* would be a formal, finite sum $s = \sum g_i 1_{\mathfrak{X}_i}$ with $g_i \in \mathbf{G}$ and \mathfrak{X}_i formal motives. However, how to interpret this as a function? As usual, we should do this at each fat point \mathfrak{w} , and interpret $1_{\mathfrak{X}}(\mathfrak{w})$ as the characteristic function on $X(\mathfrak{w})$ of $\mathfrak{X}(\mathfrak{w})$, where X is an ambient space of \mathfrak{X} . Likewise, provided X is an ambient space for all \mathfrak{X}_i , we let $s(\mathfrak{w})$ be the function $X(\mathfrak{w}) \rightarrow \mathbf{G}$ associating to a \mathfrak{w} -rational point $a \in X(\mathfrak{w})$ the sum of all g_i for which $a \in \mathfrak{X}_i(\mathfrak{w})$. However, the main obstruction is that this point-wise defined function is in general not functorial. The reason is the non-functorial nature of fibers, which in turn stems from the lack of complements in categories—note that the complement of any fiber is the union of the other fibers. To recover functoriality, we work over a subcategory of fat points:

Flat and split points. More precisely, let $\mathbb{F}\text{at}_\kappa^{\text{flat}}$ and $\mathbb{F}\text{at}_\kappa^{\text{split}}$ be the respective categories of *flat points* and *split points* over κ , whose objects are fat points over κ and whose morphisms are respectively flat and split epimorphisms. Recall that a morphism $\varphi: Y \rightarrow X$ is called a *split epimorphism* if there exists a morphism, also called a *section*, $\sigma: X \rightarrow Y$ such that $\varphi\sigma$ is the identity on X . Any split epimorphism is (faithfully) flat, so that $\mathbb{F}\text{at}_\kappa^{\text{split}}$ is a subcategory of $\mathbb{F}\text{at}_\kappa^{\text{flat}}$. Note that each structure morphism $\mathfrak{z} \rightarrow \text{Spec } \kappa$ is a split epimorphism, and by base change, so is each projection map $\mathfrak{z}\mathfrak{w} \rightarrow \mathfrak{w}$.

We will call a contravariant functor \mathfrak{X} from $\mathbb{F}\text{at}_\kappa^{\text{flat}}$ (respectively, from $\mathbb{F}\text{at}_\kappa^{\text{split}}$) to the category of sets a *flat* (respectively, a *split*) presheaf. If, moreover, we have an inclusion morphism $\mathfrak{X} \subseteq X^\circ$, where X° is the restriction of the representable functor of a κ -scheme X , we call \mathfrak{X} a *flat* (respectively, a *split*) sieve. In particular, ordinary presheafs or sieves (that is to say, defined on $\mathbb{F}\text{at}_\kappa$) when restricted to $\mathbb{F}\text{at}_\kappa^{\text{split}}$ are split—and to emphasize

this, we may call them *full* sieves—, but as the next result shows, not every split sieve is the restriction of a full sieve:

15.1. Proposition. *The complement of a schemic motif $\mathfrak{X} \subseteq X^\circ$ is a flat sieve. The complement of a formal motif $\mathfrak{X} \subseteq X^\circ$ is a split sieve.*

Proof. Any (full) schemic motif is the union of closed subsieves, and hence its complement is the intersection of complements of closed subsieves. Since the intersection of (flat or split) sieves is again a sieve, we only need to verify that the complement of a single closed subsieve $Y^\circ \subseteq X^\circ$ is a flat sieve. The only thing to show is functoriality, so let $\mathfrak{w} \rightarrow \mathfrak{z}$ be a flat morphism of fat points. We have to show that under the induced map $X(\mathfrak{z}) \rightarrow X(\mathfrak{w})$ any \mathfrak{z} -rational point a not in $Y(\mathfrak{z})$ is mapped to a point not in $Y(\mathfrak{w})$. Since fat points are affine, we may replace X by an affine open, and so assume from the start that it is affine with coordinate ring A . Let I be the ideal defining Y , and let R and S be the coordinate rings of \mathfrak{z} and \mathfrak{w} respectively. The \mathfrak{z} -rational point a corresponds to a morphism $A \rightarrow R$; it does not belong to $Y(\mathfrak{z})$ if and only if the image IR of I under $A \rightarrow R$ is non-zero. Suppose towards a contradiction that $a \in Y(\mathfrak{w})$, so that $IS = 0$. Since $R \rightarrow S$ is by assumption flat, whence faithfully flat, we must have $IR = IS \cap R = 0$, contradiction.

Assume next that \mathfrak{X} is a sub-schemic motif, that is to say, $\mathfrak{X} = \text{Im}(\varphi)$ for some morphism $\varphi: Y \rightarrow X$. Let $\lambda: \mathfrak{w} \rightarrow \mathfrak{z}$ be a split epimorphism of fat points and let $a: \mathfrak{z} \rightarrow X$ be a \mathfrak{z} -rational point outside $\text{Im}(\varphi)(\mathfrak{z})$. We have to show that the image $a \circ \lambda$ of a in $X(\mathfrak{w})$ does not lie in the image of $\varphi(\mathfrak{w})$. Towards a contradiction, assume the opposite, so that $a \circ \lambda$ factors through Y , giving rise to a commutative diagram

$$(49) \quad \begin{array}{ccc} \mathfrak{w} & \xrightarrow{b} & Y \\ \lambda \downarrow & & \downarrow \varphi \\ \mathfrak{z} & \xrightarrow{a} & X. \end{array}$$

By assumption, there exists a section $\sigma: \mathfrak{z} \rightarrow \mathfrak{w}$ so that $\lambda\sigma$ is the identity on \mathfrak{z} . Let \tilde{b} be the image of b under $Y(\sigma): Y(\mathfrak{w}) \rightarrow Y(\mathfrak{z})$, that is to say, $\tilde{b} = b \circ \sigma$. The image of \tilde{b} under $\varphi(\mathfrak{z}): Y(\mathfrak{z}) \rightarrow X(\mathfrak{z})$ is by (49) equal to

$$\varphi(\mathfrak{z})(\tilde{b}) = \varphi \circ \tilde{b} = \varphi \circ b \circ \sigma = a \circ \lambda \circ \sigma = a$$

showing that a lies in the image of $\varphi(\mathfrak{z})$, contradiction.

Lastly, assume that \mathfrak{X} is formal, so that there exists for each fat point \mathfrak{z} a sub-schemic motif $\mathfrak{Y}_{\mathfrak{z}} \subseteq \mathfrak{X}$ such that $\mathfrak{Y}_{\mathfrak{z}}(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z})$. Let $\lambda: \mathfrak{w} \rightarrow \mathfrak{z}$ be a split epimorphism. Since $\mathfrak{Y}_{\mathfrak{w}} \subseteq \mathfrak{X}$, we have $-\mathfrak{X}(\mathfrak{z}) \subseteq -\mathfrak{Y}_{\mathfrak{w}}(\mathfrak{z})$. By what we just proved, $-\mathfrak{Y}_{\mathfrak{w}}(\mathfrak{z})$ is sent under $X(\lambda): X(\mathfrak{z}) \rightarrow X(\mathfrak{w})$ inside $-\mathfrak{Y}_{\mathfrak{w}}(\mathfrak{w})$, and by construction, the latter is equal to $-\mathfrak{X}(\mathfrak{w})$. A fortiori, $-\mathfrak{X}(\mathfrak{z})$ is then sent inside $-\mathfrak{X}(\mathfrak{w})$, proving the assertion. \square

15.2. Remark. It is important to note that we may not apply this argument to an arbitrary split sieve, since a section of a split morphism is not split and hence does not induce a morphism on the rational points of the split sieve. The point in the above argument is that formal motives are presheafs on the full category of fat points, and hence any section does induce a map between their rational points.

We call any Boolean combination of closed subsieves a *flat-schemic* motif. Our aim is to define motivic sites of schemic, sub-schemic and formal motives with respect to flat and/or split points, but without changing the corresponding Grothendieck ring. A priori, there might be more morphism, that is to say, natural transformations, in this restricted context, and to circumvent this issue, we only allow morphisms that extend to true motives. More precisely, we define the *flat-schemic* motivic site $\mathcal{S}\mathcal{C}\mathcal{H}_\kappa^{\text{flat}}$, as the category with objects all flat-schemic motives, and with morphisms all natural transformations $s: \mathcal{Y} \rightarrow \mathcal{X}$ of flat schemic motives which extend to a morphism of schemic motives in the sense that there are schemic motives $\mathcal{X} \subseteq \mathcal{X}'$ and $\mathcal{Y} \subseteq \mathcal{Y}'$ and a morphism of schemic motives $s': \mathcal{Y}' \rightarrow \mathcal{X}'$ whose restriction to \mathcal{Y} is s . To not introduce unwanted isomorphisms, we moreover require that if s is injective, then so must its extension s' be. Likewise, we call any Boolean combination of sub-schemic (respectively, formal) motives a *split-sub-schemic* (respectively, a *split-formal*) motif, and we define the *split-sub-schemic* motivic site $\mathcal{S}\mathcal{U}\mathcal{B}\mathcal{S}\mathcal{C}\mathcal{H}_\kappa^{\text{split}}$ (respectively, the *split-formal* motivic site $\mathcal{F}\mathcal{O}\mathcal{R}\mathcal{M}_\kappa^{\text{split}}$), as the category with objects all split-sub-schemic (respectively, split-formal) motives, and as morphisms all natural transformations which extend to a morphism of sub-schemic (respectively, formal) motives, with injective morphisms extending to injective ones. All these sites satisfy the same properties as ordinary motivic sites, apart from being defined only over a restricted category, but have the additional property that their restriction to any scheme is a Boolean lattice. At any rate, we can define their corresponding Grothendieck rings.

15.3. Proposition. *We have equalities of Grothendieck rings, $\mathbf{Gr}(\mathcal{S}\mathcal{C}\mathcal{H}_\kappa^{\text{flat}}) = \mathbf{Gr}(\mathcal{S}\mathcal{C}\mathcal{H}_\kappa)$, $\mathbf{Gr}(\mathcal{S}\mathcal{U}\mathcal{B}\mathcal{S}\mathcal{C}\mathcal{H}_\kappa^{\text{split}}) = \mathbf{Gr}(\mathcal{S}\mathcal{U}\mathcal{B}\mathcal{S}\mathcal{C}\mathcal{H}_\kappa)$, and $\mathbf{Gr}(\mathcal{F}\mathcal{O}\mathcal{R}\mathcal{M}_\kappa^{\text{split}}) = \mathbf{Gr}(\mathcal{F}\mathcal{O}\mathcal{R}\mathcal{M}_\kappa)$.*

Moreover, for each fat point \mathfrak{z} , the arc sheaf of a flat schemic (respectively, split sub-schemic, or split formal) motif exists, and is again of that form. The induced action on the corresponding Grothendieck ring is equal to that of $\nabla_{\mathfrak{z}}(\cdot)$.

Proof. I will only give the argument for the case of most interest to us, the formal motives, and leave the remaining cases, with analogous proof, to the reader. Before we do this, let us first discuss briefly Boolean lattices. Let \mathcal{B} be a Boolean lattice. Given a finite collection of subsets $X_1, \dots, X_n \in \mathcal{B}$, and an n -tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with entries ± 1 , let X_ε be the subset given by the intersection of all X_i with $\varepsilon_i = 1$ and all $-X_i$ with $\varepsilon_i = -1$. Then any element in the Boolean sublattice $\mathcal{B}(X_1, \dots, X_n)$ of \mathcal{B} generated by X_1, \dots, X_n is a disjoint union of the X_ε . In particular, if all X_i belong to a sublattice $\mathcal{L} \subseteq \mathcal{B}$, then any element in $\mathcal{B}(X_1, \dots, X_n)$ is a disjoint union of sets of the form $C - D$ with $D \subseteq C$ in \mathcal{L} .

We now define a map γ from the free Abelian group $\mathbb{Z}[\mathcal{F}\mathcal{O}\mathcal{R}\mathcal{M}_\kappa^{\text{split}}]$ to $\mathbf{Gr}(\mathcal{F}\mathcal{O}\mathcal{R}\mathcal{M}_\kappa)$ as follows. By the above argument, a typical element in $\mathcal{F}\mathcal{O}\mathcal{R}\mathcal{M}_\kappa^{\text{split}}$ is a disjoint union of split formal motives of the form $\mathcal{X} - \mathcal{Y}$ with $\mathcal{Y} \subseteq \mathcal{X}$ (full) formal motives. We define its γ -value to be the element $[\mathcal{X}] - [\mathcal{Y}]$. This is well-defined, for if it is also equal to a difference of motives $\tilde{\mathcal{X}} - \tilde{\mathcal{Y}}$ then one easily checks that $\mathcal{X} \cup \tilde{\mathcal{Y}} = \tilde{\mathcal{X}} \cup \mathcal{Y}$ and $\mathcal{X} \cap \tilde{\mathcal{Y}} = \tilde{\mathcal{X}} \cap \mathcal{Y}$, so that

$$[\mathcal{X}] + [\tilde{\mathcal{Y}}] = [\tilde{\mathcal{X}}] + [\mathcal{Y}].$$

We extend this to disjoint sums by taking the sum of the disjoint components, and then extend by linearity, to the entire free Abelian group. It is not hard to verify that γ preserves all scissor relations. So we next check that it preserves also isomorphism relations. We may again reduce to an isomorphism of the form $s: \mathcal{X} - \mathcal{Y} \rightarrow \tilde{\mathcal{X}} - \tilde{\mathcal{Y}}$ with $\mathcal{Y} \subseteq \mathcal{X}$ and $\tilde{\mathcal{Y}} \subseteq \tilde{\mathcal{X}}$ formal motives. By assumption, s extends to an injective morphism $s': \mathcal{X}' \rightarrow \tilde{\mathcal{X}}'$ with \mathcal{X}' and $\tilde{\mathcal{X}}'$ formal motives. Upon replacing \mathcal{X} and \mathcal{X}' with their common intersection, we may assume that they are equal. Since s' is injective, it induces an isomorphism between \mathcal{X} and

its image, as well between \mathfrak{Y} and its image. Hence $\mathfrak{X} - \mathfrak{Y} \cong s'(\mathfrak{X}) - s'(\mathfrak{Y})$ and, since s' extends s , the latter must be equal to $\tilde{\mathfrak{X}} - \tilde{\mathfrak{Y}}$, yielding

$$\gamma(\mathfrak{X} - \mathfrak{Y}) = [\mathfrak{X}] - [\mathfrak{Y}] = [s'(\mathfrak{X})] - [s'(\mathfrak{Y})] = \gamma(\tilde{\mathfrak{X}} - \tilde{\mathfrak{Y}})$$

as we wanted to show. Hence, γ induces a map $\mathbf{Gr}(\mathbf{Form}_\kappa^{\text{split}}) \rightarrow \mathbf{Gr}(\mathbf{Form}_\kappa)$. By construction, it is surjective and the identity on $\mathbf{Gr}(\mathbf{Form}_\kappa)$ (when viewing a full motif as a split motif), showing that it is a bijection. By construction it is also additive, and the reader readily verifies that it preserves products, thus showing that it is an isomorphism.

For the last assertion, it suffices once more to verify this on a split formal motif of the form $\mathfrak{X} - \mathfrak{Y}$ and we set

$$\nabla_{\mathfrak{z}}(\mathfrak{X} - \mathfrak{Y}) := \nabla_{\mathfrak{z}}\mathfrak{X} - \nabla_{\mathfrak{z}}\mathfrak{Y}$$

Since the \mathfrak{w} -rational points of both sides are the same, to wit, $\mathfrak{X}(\mathfrak{z}\mathfrak{w}) - \mathfrak{Y}(\mathfrak{z}\mathfrak{w})$, for any fat point \mathfrak{w} , this is well-defined, and the assertion then follows from the universal property of adjunction. \square

15.4. Remark. Consider the flat schemic motif $\mathfrak{l}_3^\circ - \mathfrak{l}_2^\circ$. It has no κ -rational points, but it does have a \mathfrak{l}_3 -rational point, namely the identity morphism on \mathfrak{l}_3 . This example shows that the analogue of Lemma 7.3 does not hold for split formal motives. We do have:

15.5. Lemma. *If \mathfrak{X} is a split formal motif and \mathfrak{z} a fat point such that $\nabla_{\mathfrak{z}}\mathfrak{X}$ is empty, then \mathfrak{X} too is empty. In particular, all arc maps are injective on each ambient space.*

Proof. We may again reduce to the case that \mathfrak{X} is of the form $\mathfrak{Y} - \mathfrak{Z}$ with $\mathfrak{Z} \subseteq \mathfrak{Y}$ (full) formal motives. Let \mathfrak{w} be an arbitrary fat point. The closed immersion $\mathfrak{w} \subseteq \mathfrak{z}\mathfrak{w}$ induces maps $\mathfrak{Z}(\mathfrak{z}\mathfrak{w}) \rightarrow \mathfrak{Z}(\mathfrak{w})$ and $\mathfrak{Y}(\mathfrak{z}\mathfrak{w}) \rightarrow \mathfrak{Y}(\mathfrak{w})$. Since composing the closed immersion with the (split) projection $\mathfrak{z}\mathfrak{w} \rightarrow \mathfrak{w}$ is the identity, the two above maps are surjective. Since $\nabla_{\mathfrak{z}}\mathfrak{X}$ is the empty motif, it has no \mathfrak{w} -rational points, that is to say, $\mathfrak{Z}(\mathfrak{z}\mathfrak{w}) = \mathfrak{Y}(\mathfrak{z}\mathfrak{w})$. Surjectivity then yields that $\mathfrak{Z}(\mathfrak{w}) = \mathfrak{Y}(\mathfrak{w})$, whence $\mathfrak{X}(\mathfrak{w}) = \emptyset$. Since this holds for any fat point \mathfrak{w} , we see that \mathfrak{X} is the empty motif.

To prove the last assertion, assume $\nabla_{\mathfrak{z}}\mathfrak{X} = \nabla_{\mathfrak{z}}\mathfrak{Y}$ for $\mathfrak{X}, \mathfrak{Y}$ split formal motives on a scheme X . By what we just proved, $\mathfrak{X} - (\mathfrak{X} \cap \mathfrak{Y})$ and $\mathfrak{Y} - (\mathfrak{X} \cap \mathfrak{Y})$ are both empty, from which the claim now follows. \square

From now on, we will work in the largest of these motivic sites, the category of split-formal motives $\mathbf{Form}_\kappa^{\text{split}}$, and we view the class of any such motif as an element in the localized Grothendieck ring $\mathbf{G} := \mathbf{Gr}(\mathbf{Form}_\kappa)_\mathbb{L}$. Let $\underline{\mathbf{G}}$ be the *constant presheaf with values in \mathbf{G}* , that is to say, the contravariant functor on the category of split points which associates to any fat point the set \mathbf{G} and to any split epimorphism of fat points the identity on \mathbf{G} . Given a morphism, that is to say, a natural transformation, $s: \mathfrak{X} \rightarrow \underline{\mathbf{G}}$, we define, for each $g \in \mathbf{G}$, the *fiber* $s^{-1}(g)$ as the subfunctor of \mathfrak{X} given at each fat point \mathfrak{z} by the fiber $s(\mathfrak{z})^{-1}(g)$ of $s(\mathfrak{z}): \mathfrak{X}(\mathfrak{z}) \rightarrow \mathbf{G}$ at g . If both \mathfrak{X} and all fibers are split formal motifs, and s has only finitely many non-empty fibers, then we call s a *formal invariant*.

15.6. Corollary. *The formal invariants on a split formal motif \mathfrak{X} form an algebra over \mathbf{G} .*

Proof. Clearly, any multiple of a formal invariant by an element in \mathbf{G} is again a formal invariant. Let $s, t: \mathfrak{X} \rightarrow \underline{\mathbf{G}}$ be formal invariants. We have to show that $s + t$ and st are also formal invariants. Functoriality is easily verified, so we only need to show that the fibers are again split formal motifs. Fix a fat point \mathfrak{z} , and an element $g \in \mathbf{G}$. A \mathfrak{z} -rational point $a \in \mathfrak{X}(\mathfrak{z})$ lies in $(s + t)^{-1}(g)(\mathfrak{z})$ (respectively, in $(st)^{-1}(g)(\mathfrak{z})$), if $s(\mathfrak{z})(a) + t(\mathfrak{z})(a) = g$ (respectively, if $s(\mathfrak{z})(a) \cdot t(\mathfrak{z})(a) = g$). Since $s(\mathfrak{z})$ and $t(\mathfrak{z})$ have finite image, their are only

finitely many ways that g can be written as a sum $p + q$ (respectively, a product pq), with p in the image of $s(\mathfrak{z})$ and q in the image of $t(\mathfrak{z})$. Hence, the rational point a lies in the intersection $(s(\mathfrak{z})^{-1}(p)) \cap (t(\mathfrak{z})^{-1}(q))$, for one of these finitely many choices of p and q . Since a finite union of intersections of split formal motives is again split formal, the result follows. \square

Motivic integrals. Let X be a κ -scheme, \mathfrak{z} a fat point, and $s: \mathfrak{X} \rightarrow \underline{\mathbf{G}}$ a formal invariant with \mathfrak{X} a split formal motif on X . We define the (*split*) *motivic integral* of s on X along \mathfrak{z} as

$$(50) \quad \int s d_{\mathfrak{z}} X := \mathbb{L}^{-dl} \sum_{g \in \mathbf{G}} g \cdot [\nabla_{\mathfrak{z}}(s^{-1}(g))],$$

where d is the dimension of X and l the length of \mathfrak{z} . Note that the sum on the right hand side of (50) is finite by definition, so that $\int s d_{\mathfrak{z}} X$ is a well-defined element in \mathbf{G} . At the reduced fat point, $\text{Spec } \kappa$, we drop the subscript in the measure, and so the this integral becomes

$$\int s dX := \mathbb{L}^{-d} \sum_{g \in \mathbf{G}} g \cdot [s^{-1}(g)].$$

To a formal motif \mathfrak{Y} on X , we can associate two invariants. Firstly, the constant map, denoted again \mathfrak{Y} , which at each fat point is the constant map sending every rational point to $[\mathfrak{Y}]$. One easily calculates that

$$\int \mathfrak{Y} d_{\mathfrak{z}} X = [\mathfrak{Y}] \int d_{\mathfrak{z}} X = [\mathfrak{Y}] \cdot \mathbb{L}^{-dl} \cdot [\nabla_{\mathfrak{z}} X].$$

In particular, $\int dX = \mathbb{L}^{-d}[X]$ is the normalized class map. It follows from Theorem 9.7 and Proposition 10.3 that the integral $\int \mathfrak{Y} d_{\mathfrak{z}} X$ only depends on the classes of \mathfrak{Y} and X . Moreover, by our previous discussion $\int d_{\mathfrak{z}} X$ has positive dimension.

Secondly, we define the *characteristic function* $\mathbf{1}_{\mathfrak{Y}}$ of \mathfrak{Y} by the rule that $\mathbf{1}_{\mathfrak{Y}}(\mathfrak{z})$ is the characteristic function of $\mathfrak{Y}(\mathfrak{z})$, that is to say, the map sending a rational point $a \in X(\mathfrak{z})$ to 1, if $a \in \mathfrak{Y}(\mathfrak{z})$, and to zero otherwise, for any fat point \mathfrak{z} . By Proposition 15.1, this is a formal invariant. Moreover, any formal invariant can be written as a \mathbf{G} -linear combination of characteristic functions, and, in fact, the decomposition

$$(51) \quad s = \sum_{i=1}^n g_i \mathbf{1}_{\mathfrak{Y}_i}$$

is unique if the non-empty formal submotives \mathfrak{Y}_i are mutually disjoint (note that then necessarily $\mathfrak{Y}_i = s^{-1}(g_i)$), and is called the *fiber decomposition* of s .

Since $\mathbf{1}_{\mathfrak{Y}}^{-1}(1) = \mathfrak{Y}$, we get

$$(52) \quad \int \mathbf{1}_{\mathfrak{Y}} d_{\mathfrak{z}} X = \mathbb{L}^{-dl} \cdot [\nabla_{\mathfrak{z}} \mathfrak{Y}]$$

Using common practice, we will write

$$\int s d_{\mathfrak{z}} X := \int s \cdot \mathbf{1}_{\mathfrak{X}} d_{\mathfrak{z}} X.$$

In this notation, we have

$$\int s d_{\mathfrak{z}} X = \sum_{g \in \mathbf{G}} g \int_{s^{-1}(g)} d_{\mathfrak{z}} X$$

15.7. Proposition. *For each κ -scheme X and each fat point \mathfrak{z} , the motivic integral on X along \mathfrak{z} is a \mathbf{G} -linear functional on the \mathbf{G} -algebra of formal invariants.*

Proof. Motivic integration is clearly preserved under multiplication by a constant $g \in \mathbf{G}$. To prove additivity, we may induct on the number of characteristic functions, and reduce to the case of a sum $s + h\mathbf{1}_{\mathfrak{Z}}$, that is to say, we have to prove

$$(53) \quad \int s + h\mathbf{1}_{\mathfrak{Z}} d_3 X = \int s d_3 X + \int h\mathbf{1}_{\mathfrak{Z}} d_3 X.$$

Let (51) be the fiber decomposition of s . Since the fiber decomposition of $s + h\mathbf{1}_{\mathfrak{Z}}$ is then

$$\sum_{i=1}^n g_i \mathbf{1}_{\mathfrak{Y}_i - \mathfrak{Z}} + \sum_{i=1}^n (g_i + h) \mathbf{1}_{\mathfrak{Y}_i \cap \mathfrak{Z}} + h\mathbf{1}_{\mathfrak{Z} - \mathfrak{Y}}$$

where \mathfrak{Y} is the union of the \mathfrak{Y}_i , the left hand side of (53) is

$$\mathbb{L}^{-dl} \left(\sum_{i=1}^n g_i [\nabla_{\mathfrak{Z}}(\mathfrak{Y}_i - \mathfrak{Z})] + \sum_{i=1}^n (g_i + h) [\nabla_{\mathfrak{Z}}(\mathfrak{Y}_i \cap \mathfrak{Z})] + h[\nabla_{\mathfrak{Z}}(\mathfrak{Z} - \mathfrak{Y})] \right),$$

where d and l are respectively the dimension of X and the length of \mathfrak{Z} . Grouping together the $n + 1$ terms with coefficient h , and for each i , the two terms with coefficient g_i , this sum becomes

$$\mathbb{L}^{-dl} \left(\sum_{i=1}^n g_i [\nabla_{\mathfrak{Z}} \mathfrak{Y}_i] + h[\nabla_{\mathfrak{Z}} \mathfrak{Z}] \right),$$

since $\nabla_{\mathfrak{Z}}$ acts on the Grothendieck ring by Theorem 9.7, and since both $\mathfrak{Y}_i - \mathfrak{Z}$ and $\mathfrak{Z} - \mathfrak{Y}$ are disjoint from $\mathfrak{Y}_i \cap \mathfrak{Z}$. However, this is just the right hand side of (53), and so we are done. \square

Let $s: \mathfrak{X} \rightarrow \mathbf{G}$ be a formal invariant on a κ -scheme X . Given an open $U \subseteq X$, let $s|_U$ denote the restriction of s to $\mathfrak{X} \cap U^\circ$. It is easy to see that $s|_U$ is a formal invariant on U . Let U_1, \dots, U_n be an open covering of X . For each non-empty subset $I \subseteq \{1, \dots, n\}$, let U_I be the intersection of all U_i with $i \in I$. We have the following local formula for the motivic integral (here we call a scheme *equidimensional* if every non-empty open has the same dimension as the scheme):

15.8. Theorem. *Let $s: \mathfrak{X} \rightarrow \mathbf{G}$ be a formal invariant on an equidimensional κ -scheme X , let \mathfrak{z} be a flat point, and let U_1, \dots, U_n be an open covering of X . Then we have an equality*

$$(54) \quad \int s d_3 X = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|} \int s|_{U_I} d_3 U_I.$$

Proof. Given $g \in \mathbf{G}$, one easily verifies that we have an equality of motives

$$(s|_{U_I})^{-1}(g) = s^{-1}(g) \cap U_I^\circ,$$

for each $I \subseteq \{1, \dots, n\}$. Applying the scissor relations to this, we get an identity

$$[s^{-1}(g)] = \left[\bigcup_{i=1}^n (s|_{U_i})^{-1}(g) \right] = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|} [(s|_{U_I})^{-1}(g)]$$

in \mathbf{G} . Applying the arc morphism $\nabla_{\mathfrak{z}}$, as per Theorem 9.7, we get

$$[\nabla_{\mathfrak{z}}(s^{-1}(g))] = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|} [\nabla_{\mathfrak{z}}((s|_{U_I})^{-1}(g))].$$

Since by assumption all non-empty U_I have the same dimension as X (and, of course, the empty ones do not contribute), the result follows from (50). \square

Relations among motivic series. Given an element $\alpha \in \mathbf{Gr}_0(\mathcal{S}\mathbf{ch}_\kappa)$, we define

$$\int s d_\alpha X := \sum_{i=1}^s n_i \int s d_{\mathfrak{z}_i} X$$

where $\alpha = n_1[\mathfrak{z}_1] + \cdots + n_s[\mathfrak{z}_s]$ is the unique decomposition in classes of fat points given by Proposition 14.4. We then formally extend this over $\mathbf{Gr}_0(\mathcal{S}\mathbf{ch}_\kappa)[[t]]$, by treating t as a constant. In this sense, we get, for a closed germ (Y, P) , and a κ -scheme X , the following identity of power series:

$$\mathrm{Igu}_{X^{\mathrm{mot}}}^{(Y, P)} = \int d_{\mathrm{Hilb}_{(Y, P)}^{\mathrm{mot}}} X.$$

16. APPENDIX: LATTICE RINGS

Let \mathbb{M} be a motivic site over an algebraically closed field κ and let X be an κ -scheme. By assumption, $\mathbb{M}|_X$ is a lattice, and so we can define its *lattice group* $\Lambda^X(\mathbb{M})$ as the free Abelian group on \mathbb{M} -motives on X modulo the scissor relations

$$\langle \mathfrak{X} \rangle + \langle \mathfrak{Y} \rangle - \langle \mathfrak{X} \cup \mathfrak{Y} \rangle - \langle \mathfrak{X} \cap \mathfrak{Y} \rangle$$

for any two \mathbb{M} -motives \mathfrak{X} and \mathfrak{Y} on X . In other words, same definition as for the Grothendieck ring, but without the isomorphism relations. In particular, there is a natural linear map $\Lambda^X(\mathbb{M}) \rightarrow \mathbf{Gr}(\mathbb{M})$. We will denote the class of a motif \mathfrak{X} again by $[\mathfrak{X}]$. For each n , consider the embedding $\mathbb{M}|_{X^n} \rightarrow \mathbb{M}|_{X^{n+1}}$ via the rule $\mathfrak{X} \mapsto \mathfrak{X} \times X^\circ$. One verifies that this induces a well-defined linear map $\Lambda_n := \Lambda^{X^n}(\mathbb{M}) \rightarrow \Lambda_{n+1} := \Lambda^{X^{n+1}}(\mathbb{M})$, where X^n is the n -fold Cartesian power of X . Moreover, the Cartesian product defines a multiplication $\Lambda_m \times \Lambda_n \rightarrow \Lambda_{m+n}$, for all m, n . Hence $\bigoplus_n \Lambda_n$ is a graded ring, called the *graded lattice ring* of \mathbb{M} on X , and denoted $\Lambda_\bullet^X(\mathbb{M})$. It admits a natural ring homomorphism into $\mathbf{Gr}(\mathbb{M})$.

We can now state a combinatorial property of the split motivic integral:

16.1. Proposition. *Over a κ -scheme X and a fat point \mathfrak{z} , we can define for each formal invariant $s: \mathfrak{X} \rightarrow \underline{\mathbf{G}}$ on X and each $g \in \Lambda^X(\mathbf{Form}_\kappa)$, an integral $\int_g s d_{\mathfrak{z}} X$, such that if g is the class in $\Lambda^X(\mathbf{Form}_\kappa)$ of a formal motif \mathfrak{Y} on X , then*

$$\int_g s d_{\mathfrak{z}} X = \int_{\mathfrak{Y}} s d_{\mathfrak{z}} X.$$

Proof. By definition, g is a \mathbb{Z} -linear combination of classes of formal motives on X , say, of the form $g = n_1[\mathfrak{Y}_1] + \cdots + n_s[\mathfrak{Y}_s]$. Define

$$\int_g s d_{\mathfrak{z}} X := \sum_{i=1}^s n_i \int_{\mathfrak{Y}_i} s d_{\mathfrak{z}} X.$$

To show that this is well-defined, we have to verify this only for scissor relations, that is to say, we have to show that

$$\int_{\mathfrak{Y}} s d_{\mathfrak{z}} X + \int_{\mathfrak{Y}'} s d_{\mathfrak{z}} X = \int_{\mathfrak{Y} \cup \mathfrak{Y}'} s d_{\mathfrak{z}} X + \int_{\mathfrak{Y} \cap \mathfrak{Y}'} s d_{\mathfrak{z}} X$$

for $\mathfrak{Y}, \mathfrak{Y}'$ formal motives on X . This is immediate from the easily proven identity of characteristic functions

$$\mathbf{1}_{\mathfrak{Y}} + \mathbf{1}_{\mathfrak{Y}'} = \mathbf{1}_{\mathfrak{Y} \cup \mathfrak{Y}'} + \mathbf{1}_{\mathfrak{Y} \cap \mathfrak{Y}'}.$$

□

Using this, we can now show that the lattice rings are not very interesting invariants (and hence only by also taking isomorphism relations, do we get something significant):

16.2. Corollary. *The natural map sending a formal motif on some Cartesian power of X to its class in $\Lambda_{\bullet}^X(\text{Form}_{\kappa})$ is injective.*

Proof. Note that there are no non-trivial relations among classes of motives on different Cartesian powers of X , so after replacing X by one of its Cartesian powers, we may reduce to the case that \mathfrak{X} and \mathfrak{Y} are formal motives on X having the same class in $\Lambda^X(\text{Form}_{\kappa})$. By Proposition 16.1, we have

$$(55) \quad \int_{\mathfrak{X}} s \, d_3 X = \int_{\mathfrak{Y}} s \, d_3 X$$

for any formal invariant s on X and any fat point \mathfrak{z} . Take $s := 1_{\mathfrak{X}}$. The left hand side of (55) is equal to $\mathbb{L}^{-dl}[\nabla_{\mathfrak{z}} \mathfrak{X}]$ (as an element in \mathbf{G}), whereas the right hand side is equal to $\mathbb{L}^{-dl}[\nabla_{\mathfrak{z}}(\mathfrak{X} \cap \mathfrak{Y})]$, where d and l are respectively the dimension of X and the length of \mathfrak{z} . Using that $\nabla_{\mathfrak{z}}$ preserves scissor relations, we get $\nabla_{\mathfrak{z}}(\mathfrak{X} - \mathfrak{Y}) = 0$. Hence $\mathfrak{X} - \mathfrak{Y} = \emptyset$ by Lemma 15.5, showing that $\mathfrak{Y} \subseteq \mathfrak{X}$. Replacing the role of \mathfrak{X} and \mathfrak{Y} then proves the other inclusion. \square

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